ONE DIMENSIONAL SETS AND PLANAR SETS ARE ASPHERICAL

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Abstract. We give a relatively short proof of the theorem that planar sets are aspherical. The first proof of this theorem, by third author Andreas Zastrow, was considerably longer.

1. Introduction. A space K is said to be *aspherical* if each continuous function $f : \mathbf{S}^k \to K$ from a sphere $\mathbf{S}^k$ of dimension $k \geq 2$ into the space K is nullhomotopic in K. Demonstration of the "obvious" fact that all planar sets are aspherical has proved elusive. Zastrow [Z] finally succeeded in proving this fact by an exhaustive analysis of all paths in a planar set; his proof is lengthy, though elementary in the sense that it uses essentially no higher dimensional topology. We give here a shorter, though less elementary, proof that makes use of standard techniques of geometric and algebraic topology. It has been known since at least 1957 (see [CF]) that 1-dimensional separable metric spaces are aspherical. We reduce the planar theorem to this 1-dimensional case. We include our own proof of the 1-dimensional case by a method which we find intriguing and suggestive.

Since we aim to show that *all* planar sets K are aspherical, we may make an obvious first reduction, namely, to the case where $K = f(\mathbf{S}^k)$. That is, **it is both necessary and sufficient to show that f is nullhomotopic in its own image**. Since the image of f is a Peano continuum (compact, connected, locally connected and metric), we may restrict ourself to the study of Peano continua. Many wonderful facts are known about such continua that will help us in our work.

Our guiding intuition is that, in a planar Peano continuum, path classes should be represented by unique geodesics which vary continuously. Using these unique geodesics, we should simply be able to shrink the map $f : \mathbf{S}^k \to K$ "radially" along the representing geodesics to some base point.

Our actual proof realizes the guiding intuition precisely in the case of a 1-dimensional Peano continuum. Although the 1-dimensional case is already known [CF], we include our own short proof.

There are two parts to our proof in the 1-dimensional case, and these appear in Sections 2 and 3. We introduce and use the notion of path parametrization by oscillation, a notion which we find intriguing.

The guiding intuition can probably also be realized with the 2-dimensional planar Peano continuum as well. We have made considerable progress in justifying our intuition in this more general case, but we do not report those incomplete results here.

Instead, in this paper we reduce the 2-dimensional case to the 1-dimensional case by the intuitively simple technique of pushing enough holes into the 2-dimensional image of f that $f(\mathbf{S}^k)$ becomes 1-dimensional in the limit. The details appear in Section 4. (We are using here the well-known fact that a compactum K in the plane has Brouwer covering dimension ≤ 1 iff it contains no open subset of the plane. Indeed, one can use the holes in K as places from which one can push K by a small map into the 1-skeleton $T^{(1))}$ of a fine triangulation of the plane. The 1-skeleton has a small covering with one dimensional nerve. This covering pulls back to a fine covering of K with 1-dimensional nerve. Hence K is 1-dimensional.)

In the process of pushing holes into $f(\mathbf{S}^k)$, we are not able to make the image of $\mathbf{S}^k$ nonseparating and we are not able to maintain the homotopy type of $f(\mathbf{S}^k)$. The process of pushing holes into K is more delicate than one might imagine since there are Peano continua in the plane that are not homotopy equivalent to 1-dimensional continua; Zastrow [Z] suggested a possible example, and we prove in Section 5 that this

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suggested example is correct.

We thank the referee for a careful reading of this very technical paper and for good suggestions to improve the exposition. In particular, we have followed the suggestion that we disentangle the various parts of the induction from one another to make the proof easier to understand.

2. Parametrizing paths by oscillation. The standard method of parametrizing paths is by arc length. We present here an alternative parametrization that makes every path rectifiable. The idea is simply to count, continuously, the number of times a path oscillates between parallel planes. We shall eventually need to consider a dense set of such plane pairs, but here is a description of the oscillation count for a single pair.

In Euclidean space $\mathbf{E}^n$, let (P, Q) denote a pair of parallel $(n-1)$-planes. Let $g_{(P,Q)} : \mathbf{E}^n \to [-1, 1]$ denote the function which takes the half space which is bounded by P, and which omits Q, to -1; which takes the half space which is bounded by Q, and which omits P, to $+1$; which takes the middle third of the slice between P and Q to 0; and which extrapolates as appropriate linearly between -1 and 0, or between 0 and 1, on the remaining two thirds.

We shall use the function $g_{(P,Q)}$ to count *continuously* the number of times a path oscillates between the planes P and Q. Suppose that $f : [a, b] \to \mathbf{E}^n$ is an arbitrary path. Consider an arbitrary finite collection of points from the interval $[a, b]$: $a \le a_0 \le a_1 \le a_2 \le \cdots \le a_k \le b$. Define the *oscillation* $o(f, (P, Q))$ of f with respect to the plane-pair (P, Q) to be the supremum over all such finite collections of points of the sum,

$$-\sum_{i=1}^{k} g_{(P,Q)}(f(a_{i-1})) \cdot g_{(P,Q)}(f(a_i)).$$

For a collection consisting of 0 points or 1 point, we take the sum to be 0, so that the supremum is always ≥ 0.

Let's explore for a moment the meaning of the sum. If the points $a_0, a_1, \ldots, a_k$ have images which lie alternately on P and Q, then the sum simply counts the number of alternations. If two successive points have images whose $g_{(P,Q)}$ values have the same sign, then the sum can be increased by deleting that one of the two points whose $g_{(P,Q)}$ value has smallest absolute value. The only way in which a pair of successive points can contribute a positive increment to the sum is to have one of the two image points near P, the other near Q. Since the path can only cross the middle third of the slice between P and Q a finite number of times, there is an integer K such that for all $k > K$, the sum can be increased by deleting an appropriate subset of the points a_i. From such considerations we easily obtain the following result.

2.1. Lemma. The three parameters f, P, and Q can, of course, all be parametrized by the compact open topology. With respect to those topologies, the function $o(f, (P, Q))$ is a continuous function of all three parameters f, P, and Q. There is a collection of points a_i for which the sum realizes the supremum.

Proof. We may pick the integer K mentioned above so large that for all f', P', and Q' near to f, P, and Q, the path f' cannot cross the middle third between P' and Q' more than K times. Hence we may restrict our considerations to sets of a_i of size $\le K$. But the collection of such sets is compact, and the corresponding sum is continuous. Thus the supremum is realized. This maximum value is obviously a continuous function on this compact set. This completes the proof of the lemma.

2.2. Definition. If we take a model plane pair, then all plane pairs may be obtained by a composition of the following functions: a dilation (to choose the proper distance between planes), a rotation (to choose the orientation of those planes), and a translation. Thus the set of plane pairs may be topologized by the compact open topology, and the resulting space is separable metric. Hence, we may choose a countable dense set $\{(P_i, Q_i)\}$ of plane pairs. Then we define the *total oscillation* of a path $p : [a, b] \to \mathbf{E}^n$ by the following formula:

$$T(p) = \sum_{i=1}^{\infty} \left(\frac{1}{2^i}\right) \left[\frac{o(p, (P_i, Q_i))}{1 + o(p, (P_i, Q_i))}\right].$$

Of course the manipulations are simply designed to make the individual oscillations summable. We leave the easy proof of the following theorem to the reader.

2.3. Theorem. The total oscillation of a path is a continuous function on the space of paths. It vanishes on a path if and only if the path is a constant path. If a path α is the concatenation of paths β and γ, with γ nonconstant, then $T(\alpha) > T(\beta)$. Thus every path α may be parametrized by total oscillation. This parametrization collapses constant subpaths to degenerate subpaths.

2.4. Contracting a path by total oscillation. If α is a path with designated first point and last point, then there is a natural contraction of α to a point defined by means of total oscillation. For each $t \in [0, \infty)$, define

$$g(s,t) = \max\{0, s-t\}.$$

Define the contraction $H_\alpha : [0, T(\alpha)] \times [0, \infty) \to \mathbf{E}^n$ by the formula

$$H_\alpha(s,t) = \alpha(g(s,t)).$$

Of course, the actual contraction is completed in finite time.

2.5. Definition. We say that a path is *geodesic* if it has minimum total oscillation in its path class.

3. Asphericity of 1-dimensional metric spaces. If $f : \mathbf{S}^k \to X$, with $k \geq 2$, is a continuous function into a 1-dimensional metric space X, then the image $K = f(\mathbf{S}^k)$ is a 1-dimensional Peano continuum. We must show that f contracts in X. It suffices to show that f contracts in K.

Every compact metric space of covering dimension $\leq k$ embeds in Euclidean $(2k+1)$-dimensional space by standard arguments in dimension theory. In particular, our 1-dimensional Peano continua embed in Euclidean 3-space. This result may help the reader to visualize our results.

We shall show that every path class in a 1-dimensional Peano continuum K contains a unique geodesic and that these geodesics vary continuously with the path class in a very strong sense. Asphericity will follow by a very simple, short argument.

Our argument depends on the analysis of paths in a 1-dimensional space which we carried out in [CC]. We review here the facts that we need from that paper. We fix for consideration a compact, pathwise connected, one-dimensional metric space X for the discussion.

3.1. Definition. A *dendrite* is a connected, locally connected, compact metric space that contains no simple closed curve. Every dendrite is contractible.

3.2. Theorem A [CC]. Let $f : S^1 \to X$ be a nullhomotopic loop. Then f factors through a surjective map $f' : S^1 \to D$, where D is a planar dendrite.

3.3. Theorem B (Combinatorial description of the dendrite D of Theorem A) [CC]. For each $x \in D$, let H_x denote the convex hull of ${f'}^{-1}(x)$ in B^2. If $x \neq y$, then $H_x \cap H_y = \emptyset$. Then the dendrite D may be realized as the cellular upper semicontinuous decomposition $D = B^2/\{H_x \,|\, x \in D\}$ of B^2. That is, one can analyze the structure of D completely in terms of the geometry of the decomposition $\{H_x \,|\, x \in D\}$ of B^2.

3.4. Definition. We say that a nonconstant loop $f : (S^1, 1) \to (X, x_0)$ is *reducible* if there is an open arc $I = (x, y)$ in $S^1 \setminus \{1\}$ such that $f(x) = f(y)$ and the resulting loop based at $f(x) = f(y)$ defined by $f\,|\,[x, y]$ is nullhomotopic. We say f is *reduced* if it is not reducible. A constant loop is, by definition, *reduced*. If one allows the open arc (x, y) to include the base point 1, then one obtains a notion of *cyclically reducible* and the corresponding notion of *cyclically reduced* loop.

3.5. Theorem C [CC]. Every loop $f : (S^1, 1) \to (X, x_0)$ is homotopic to a reduced loop. This reduced loop is unique up to reparametrization of S^1. Similarly, every loop is freely homotopic to a cyclically reduced loop, and that loop is unique up to cyclic reparametrization.

3.6. Theorem. Suppose that K is a 1-dimensional Peano continuum. Then, in each path class in K there is a unique geodesic.

Proof. Let p be a path which is not reduced, and let p' denote the corresponding reduced path (Theorem C). Let f denote the loop formed by concatenating p and the reverse of p'. Then f is nullhomotopic, so that f factors through a surjective map $f' : S^1 \to D$, where D is a planar dendrite (Theorem A). The dendrite

D and the map f' can be analyzed as in Theorem B. Consider the sets H_x which intersect the domain of p' in S^1. Since p' is reduced, each must intersect the domain of p' in only a single point. The union of these H_x is a closed subset of B^2. If we subtract this closed set from S^1, the components of the remaining set form a null sequence of open arcs, each of which represents a reduction of p. That is, p may be reduced to p' by a null sequence of reductions. In particular, unless p is already constant on each of these subarcs, $T(p') < T(p)$. We conclude that p' is the unique geodesic in the path class of p. This completes the proof of the theorem.

We now wish to show that geodesics vary continuously in a very strong sense.

3.7. Definition. We say that paths α and β in K are *δ-end-point-homotopic* if there is a homotopy from α to β in K such that the paths joining the endpoints of α and β in the homotopy each have diameter less than δ.

3.8. Definition. We say that the parametrizations of paths α and β by total oscillation are *ϵ-close*, where $\epsilon > 0$, if, for each $t \in [0, \infty)$, $H_\alpha(T(\alpha), t)$ and $H_\beta(T(\beta), t)$ lie within ϵ of one another, where the homotopies considered are the contractions given by total oscillation.

3.9. Theorem. Suppose that K is a 1-dimensional Peano continuum. Then, for each $\epsilon > 0$, there is a $\delta > 0$ such that, if α' and β' are paths which are δ-end-point-homotopic in K, and, if α and β are the corresponding geodesics, then the parametrizations of α and β by total oscillation are ϵ-close.

Proof. Because α' and β' are δ-end-point-homotopic in K, the same is true of the corresponding geodesics α and β. Thus we may form a null-homotopic loop f as a concatenation of α, a δ-path a, the reverse of β, and a δ-path b. We lose no generality in assuming that a and b are also geodesic.

By Theorem A, the loop f factors through a surjective map $f' : S^1 \to D$, where D is a planar dendrite. We may use Theorem B to analyze the map f'. Each of the convex hulls H_x can intersect any given one of the four geodesic boundary arc domains in at most one point. If one of these sets H_x intersects both the domain of a and the domain of b, then β and α both are homotopic to paths in K of diameter $< 2\delta$. But, by an easy argument, for δ sufficiently small, this would imply that the diameter of the set image$(\alpha) \cup$image(β) is $< \epsilon$, from which the desired result would follow. Otherwise, it is easy to analyze the exact form of the H_x's: there would be precisely two H_x's which intersect S^1 in more than 2 points, one of which intersects the domains of α, a, and β, the other of which intersects the domains of α, b, and β. All other elements would join precisely two of the four domain arcs. The situation is described exactly in Figure 1. It is an exercise to show that there is a $\delta > 0$ such that any arc of diameter $< \delta$ has total oscillation less than $\epsilon/2$. Using the contraction by total oscillation defined above, it is easy to see that the parametrizations of α and β by total oscillation are ϵ-close.

3.10. Theorem. Suppose that X is a 1-dimensional metric space. Then X is aspherical.

Proof. Suppose that $k \geq 2$ and that $f : \mathbf{S}^k \to X$ is a continuous function. We restrict our attention to $K = f(\mathbf{S}^k)$. Fix a base point x_0 in $\mathbf{S}^k$. For each $x \in \mathbf{S}^k$, there is a unique path class p_x in $\mathbf{S}^k$ joining x_0 to x. Let q_x denote the geodesic in K corresponding to the image of p_x. Contract f by contracting each q_x by means of the parametrization by total oscillation. Since, for nearby x's, the paths q_x are homotopic by homotopies which do not move endpoints very far, it follows from the previous theorem that the contraction is continuous. This completes the proof.

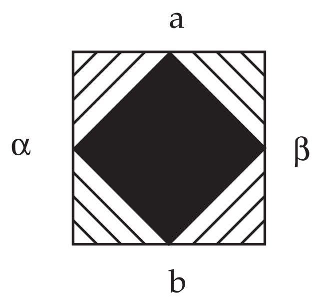
4. Asphericity of planar sets.

4.1. Theorem. Every planar set is aspherical.

We will prove three lemmas and the implications, Lemma 3 $\Rightarrow$ Lemma 2 $\Rightarrow$ Lemma 1 $\Rightarrow$ Theorem. Lemma 3 contains the heart of the matter, and its proof will be complicated.

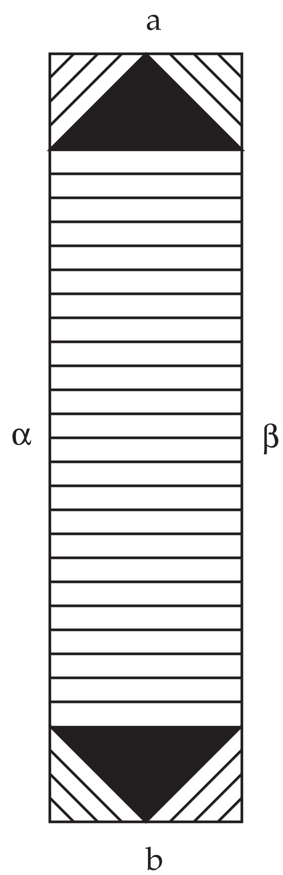
4.2. Lemma 1. Suppose that $f : \mathbf{S}^k \to \mathbf{E}^2$ is a continuous function, with $k \geq 2$. Then f is homotopic within its own image to a map $g : \mathbf{S}^k \to \mathbf{E}^2$ which has 1-dimensional image.

4.3. Lemma 2. Suppose that K is a planar Peano continuum and that $f : \mathbf{S}^k \to K$ is a continuous function, that n is an integer (possibly negative or 0), and that every point of K lies within $1/2^n$ of $\mathbf{E}^2 \setminus K$. Then f is homotopic within K to a map f' by a homotopy which moves no point further than $1/2^n$ such



Case 1. Some convex hull intersects both a and b .

Case 2. No convex hull intersects both a and b.



Analysis of the dendrite D by means of the convex hulls H_x.

that every point of the image of f' lies within $1/2^{n+1}$ of the complement of that image.

4.4. Lemma 3. Suppose that K is a planar Peano continuum and that $f : \mathbf{S}^k \to K$ is a continuous function. Suppose that K has nonempty interior in $\mathbf{E}^2$ and that B is a closed disk contained in that interior. Finally, suppose that N is a connected open neighborhood of B in K which intersects ∂K. Then f is homotopic in K to a map f' whose image misses B by a homotopy which moves points only in N.

Lemma 1 $\Rightarrow$ Theorem: Since 1-dimensional Peano continua are aspherical by the results of the last section, the map g is nullhomotopic in its own image. Hence f is nullhomotopic in K, and K is therefore aspherical.

Lemma 2 $\Rightarrow$ Lemma 1: Let $K_0 = K$. Let $f_0 = f$. Choose an integer n_0 so small (possibly large negative) that K_0 has diameter $< 1/2^{n_0}$.

Proceed inductively. By Lemma 2, the map f_i is homotopic within K_i to a map f_{i+1} by a homotopy which moves no point further than $1/2^{n_i}$ such that every point of the image K_{i+1} of f_{i+1} lies within $1/2^{n_{i+1}}$ of the complement of K_{i+1}, where $n_{i+1} = n_i + 1$.

The sequences of maps and homotopies are Cauchy sequences, hence converge to a map and a homotopy. The final image is clearly 1-dimensional.

Lemma 3 $\Rightarrow$ Lemma 2: Pick a finite number of points $\{p_i\}$ in the interior of K such that every point of K is within $1/2^{n+1}$ of either the complement of K or one of these points. Pick arcs A_i of minimal possible length joining these points to ∂K. For each i, j, either the arcs A_i are disjoint or one contains the other. For each maximal A_i, pick a disk B_i in the interior of K which contains in its interior all of the points p_j which lie on A_i. Let N_i denote a small neighborhood of the union of A_i and B_i. We may make all of our choices in such a way that the neighborhoods N_i are disjoint, and each has diameter $< 1/2^n$. By Lemma 3, we may homotop f off of each of the sets B_i. Lemma 2 follows.

Finally we move to the difficult proof of Lemma 3.

Proof (Lemma 3).

Four-step outline of the proof of Lemma 3.

Preliminaries. Since every connected open subset of a Peano continuum is arcwise connected, we find the existence of an arc A in N irreducibly joining the center p of B to the boundary of K. We may assume that A is straight and B is round. That being so, we see that there is in fact a whole uncountable family of such straight arcs A in N emanating from p radially, irreducible from p to ∂K, each intersecting B in a radius. In time we shall show that uncountably many of these arcs A have certain nice relationships with $f(S^k)$. See Figure 2.

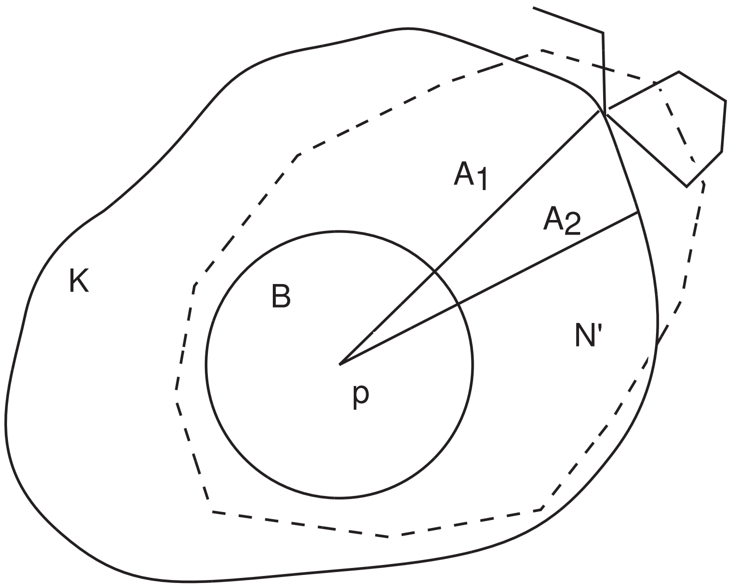


Figure 2. Two paths, A_1 and A_2, from the center p of B to $\partial(K)$.

We certainly lose no generality in assuming that, over $\mathrm{int}(K)$, the map f is piecewise linear. In particular, over almost all points and almost all lines or segments in $\mathrm{int}(K)$, the map f will be transverse so that the inverse of that line or segment will be a manifold of the appropriate codimension (point: codimension 2; line: codimension 1) with a trivial normal bundle which maps in a fiber preserving fashion onto the normal bundle of the point or line.

Step 1. We may assume that the map is regular over the center point p of B so that $P = f^{-1}(p) \subset \mathbf{S}^k$ is a $(k-2)$-manifold with tubular neighborhood $P \times B^2$ which maps in a fiber preserving fashion onto the normal bundle of the point p. We may assume that this normal bundle of the point p contains the disk B in its interior.

[Step 1 is a standard application of general position or transversality. We will therefore not expand the discussion of Step 1 below.]

Step 2. Find a $(k-1)$-dimensional manifold Q in $\mathbf{S}^k$ bounded by P such that Q has a product neighborhood $Q \times [-1, 1]$ in $\mathbf{S}^k$.

In $\mathbf{S}^k$, keeping the frontier of $Q \times [-1, 1]$ fixed and allowing a transition zone only near $P \times [-1, 1]$, homotop each $[-1, 1]$-fiber into itself so that the fiber-interval $[-1/2, 1/2]$ is collapsed to 0. See Figure 3.

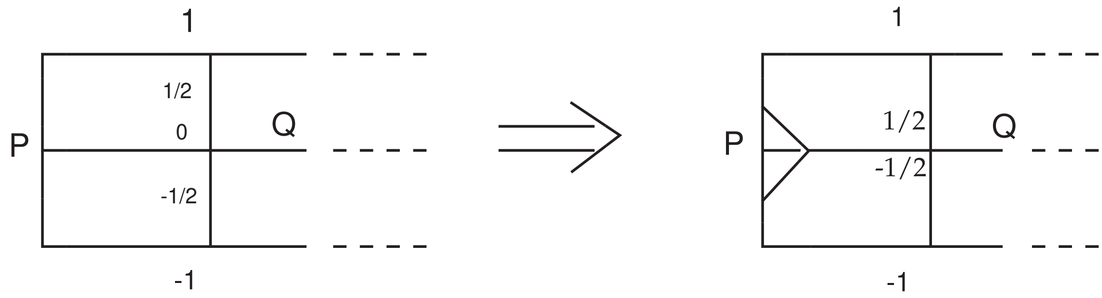


Figure 3. Collapsing normal fibers onto Q.

[Step 2 is also a standard application of general position or transversality, though not over K but over the plane $\mathbf{E}^2$: We pick a straight arc A in N as at the beginning of this proof which irreducibly joins the center point p of B to ∂K and extend slightly from the end in ∂K to form an arc A' which leads from the center point p of B to a point of the complement of K. We move the map f slightly, not in K but in $\mathbf{E}^2$, to put the map f transverse to A'. The inverse of A' will be a $(k-1)$ manifold with boundary P. We obtain the desired manifold Q by discarding the components which have empty boundary. Some particular care must be taken in the choice of A, A', and Q in order to prepare for Step 3. Hence, our discussion of Step 2 will be considerably expanded below. We will consider the family of manifolds Q which arise as we vary the arc A, the extension A', and the general position approximation to f and show by very general arguments that uncountably many of the choices are adequate for our purposes.]

Step 3. Homotop $f \,|\, Q$ within K into one of our arcs A (not the arc A used in finding Q) by a homotopy which, near B, is only a slight rotation of B.

[Step 3 contains the hardest work. This step is easiest when $k = 2$, and we will complete the proof that $\pi_2(K) = 0$ first. For $k > 2$, the proof will proceed by induction on k. Our homotopy of the manifold Q within K will, in the standard manner, be performed on a simplicial subdivision of Q cell by cell, by induction on the dimension of the cell. The most difficult part of the construction will involve the 1-skeleton. Thereafter, the inductive hypothesis that $\pi_2(L) = \cdots = \pi_{k-1}(L) = 0$ for every planar set L will be precisely what is needed to complete the homotopy.]

Step 4. For each $q \in Q$, homotop $f \,|\, \{q\} \times [-1/2, 1/2]$ along the arc A toward and across p so that the final image of the fiber misses a neighborhood of p as indicated in Figure 4. This homotopy moves f so that p is no longer in the image of f. Once p has been freed, it is an easy matter to free all of B.

[Step 4 is adequately described by the paragraph above and by Figure 4. We shall not expand the argument for Step 4 below. Step 4 completes the proof of Lemma 3.]

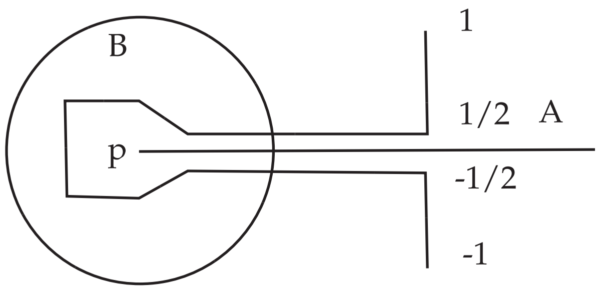


Figure 4. Pushing a normal fiber over the end of A at p.

And now we expand our discussion of Steps 2 and 3. We shall need to repeat portions of our arguments three times in slightly different settings (Cases 1, 2, and 3). Case 1 deals with dimension $k = 2$ and will complete the proof that $\pi_2(K) = 0$. Case 2 uses the same methods as Case 1 to prove a technical lemma needed for $k > 2$. Finally, Case 3 deals with the general case $k \geq 2$ by the induction on k successfully begun by Case 1. In our original exposition, we tried to combine these three cases as much as possible to avoid duplication; but readers found the exposition, with its inductive interweaving of methods, confusing. Hence, we present the cases separately.

Case 1 (k = 2). When $k = 2$, the $(k-2)$-manifold P described in Step 1 is a finite set of points. The $(k-1)$-manifold Q described in the discussion of Step 2 is a finite collection of arcs joining the points of P in pairs. Consequently, in particular, the cardinality $|P| = 2\ell$ of P is even. The idea of the proof is like that of the Hopf degree theorem which states that a map $g : \mathbf{S}^k \to \mathbf{S}^k$ is nullhomotopic if $\deg(g) = 0$; namely, we seek to homotop the map $f : \mathbf{S}^2 \to K$ along the images of the arcs of Q to remove the points of P in pairs. The details are difficult since the structure of K near $\partial(K)$ may be complicated.

Step 2.1 (Case 1). *Limits of the manifolds Q.* Take a sequence of 1-manifolds $Q = Q_i$ as described in the discussion of Step 2 above relative to shorter and shorter extensions A' of the arc A and smaller and smaller modifications of the map f.

Each collection Q_i of arcs partitions P into pairs of points. Passing to a subsequence, we may require that all of the Q_i partition P in the same way. That is, if $|P| = 2\ell$, then we may order the ℓ arcs $Q_i(1), \ldots, Q_i(\ell)$ of Q_i and assume that, for all indices a, b, c, $\partial(Q_a(c)) = \partial(Q_b(c))$. Since the space $C(\mathbf{S}^2)$ of compact subsets of $\mathbf{S}^2$ is a compact metric space with the Hausdorff metric, we may assume that, for $j = 1, \ldots, \ell$, the sequence $Q_i(j)$ converges to a compact subset $Q(j)$ of $f^{-1}(A)$. Since each of the sets $Q_i(j)$ is connected, it follows that $Q(j)$ is also connected.

Step 2.2 (Case 1). *Varying the arc A.* We now think of the arc A as a parameter which can be varied, and we insert this parameter into all of our Q notation. Our aim is to show that the ℓ continua $Q(A,1), \ldots, Q(A,\ell)$ behave enough like arcs to be useful to us. For some choices of the arc A, this may not be the case. We shall show that for many choices of the arc A the corresponding sets $Q(A,j)$ are useful.

By rotation about the center p of the disk B, we find that we may consider not only A but also uncountably many arcs A. Carrying out the construction with each of them, we find that there is an uncountable interval of suitable arcs A in N. By passing to successive subfamilies of this interval of arcs, we may restrict the properties of the associated continua $Q(A,j)$ in three ways, as follows.

(1) Since there are only finitely many possible partitions of P into pairs, uncountably many arcs A give rise to the same partition of P. Hence we may consider only those sharing a common partition of P.

(2) The following fact shows that for only countably many arcs A is it possible that some $Q(A, j(A))$ separates $\mathbf{S}^2$. Hence we may consider only those arcs A for which each $Q(A,j)$ fails to separate $\mathbf{S}^2$.

Fact. For only countably many arcs A does there exist an index $j(A)$ such that the continuum $Q(A, j(A))$ separates $\mathbf{S}^2$.

Proof (Fact). Suppose otherwise. Pick an uncountable family of arcs A for which there is an index $j(A)$ such that $Q(A, j(A))$ separates $\mathbf{S}^2$. Since the disk $B \subset K$ has only finitely many preimages in $\mathbf{S}^2$, we may assume that every one of our continua $Q(A, j(A))$ joins the centers of the same two preimages B_1 and B_2 of B. Each such $Q(A, j(A))$ intersects each of B_1 and B_2 in a single radius. We delete $\mathrm{int}(B_1)$ and $\mathrm{int}(B_2)$ from each $Q(A, j(A))$ to obtain disjoint separating subcontinua $C(A, j(A))$ of $\mathbf{S}^2$.

There is a countable dense set of point pairs in $\mathbf{S}^2$. Hence uncountably many of the continua $C(A, j(A))$ separate the same two points α and β of $\mathbf{S}^2$. Take three of these uncountably many continua separating α and β, say C_1, C_2, C_3. There is an arc $A(\alpha)$ which irreducibly joins α to $C_1 \cup C_2 \cup C_3$ and another arc $A(\beta)$ which irreducibly joins β to $C_1 \cup C_2 \cup C_3$. The union $A(\alpha) \cup A(\beta)$ misses at least one of the continua, say C_3. Then $A(\alpha) \cup A(\beta) \cup C_1 \cup C_2 \cup \mathrm{int}(B_1)$ is a connected set which misses C_3 but joins α to β, a contradiction. Hence only countably many of the $Q(A, j(A))$ can separate $\mathbf{S}^2$, as claimed.

(3) As noted above, we need only consider arcs A that intersect a given interval of $\partial(B)$. Hence the points $A \cap \partial(B)$ can be linearly ordered by angle on the circle $\partial(B)$. It is well-known, and easy to prove, that in any uncountable subset of the circle, all but countably many points are limit points of uncountable order from each side. We need such a limit property not only for the points of $A \cap \partial(B)$ but also for the corresponding continua $C(A,j)$. Thus we claim that all but countably many of the sets $Q(A) = \{Q(A,1), \ldots, Q(A,\ell)\}$ of continua are the uncountable limit of such sets of continua from both sides, in the precise sense which we shall now explain:

We may calculate limits of families of compacta in $\mathbf{S}^2$ by considering $Q(A)$, an ℓ-tuple of continua, as a point of the product space $C(\mathbf{S}^2) \times \cdots C(\mathbf{S}^2)$ (ℓ copies). This space is known to be a compact metric space and therefore has a countable basis.

The arcs A are ordered linearly by angle about the center of B. This ordering allows us to talk about whether A_1 is to the left (counterclockwise) of A_2 or to the right (clockwise) of A_2. We say that $Q(A_1)$ is to the right or left of $Q(A_2)$ if the same is true of A_1 and A_2.

We say that $Q(A_1)$ is an uncountable limit from both sides if each basic open set O containing $Q(A_1)$ also contains uncountably many of the sets $Q(A_2)$ which lie to the left of $Q(A_1)$ as well as uncountably many which lie to the right of $Q(A_1)$.

Here is the proof of our claim that most of the sets $Q(A)$ are uncountable limits both from the left and from the right:

Suppose that there are uncountably many of the $Q(A)$ that are not uncountable limit points from the left; we consider only those. Given any uncountable subset of a separable metric space, all but countably many of them are uncountable limit points of the set. If we delete the countably many exceptions, every point remaining is an uncountable limit point. Since none is an uncountable limit point from the left, all are uncountable limit points from the right. For each, from a fixed countable basis, we pick a basic open set $O(A)$ containing it that contains only countably many elements from the left. Since the basis from which we picked is countable, there is a single basic open set that is picked for each of uncountably many $Q(A)$. In particular, one of these, say $Q(A_1)$, has another, say $Q(A_2)$, to its left in the same basic open set. Since $Q(A_2)$ is an uncountable limit, uncountably many $Q(A)$'s are to the left of $Q(A_1)$ in this basic open set, a contradiction. We conclude that all but countably many $Q(A)$ are indeed uncountable limits from the left, likewise from the right.

Step 3 (Case 1). Having restricted our family of arcs in the three ways just described, any of the (uncountably many) remaining arcs A is suitable for our purposes. For such an A, since $Q(A)$ is a limit point from both sides, we may choose A_- and A_+ to the left and right of A so that $Q(A_-)$ and $Q(A_+)$ are arbitrarily close to $Q(A)$ in the Hausdorff topology. We approximate $Q(A_-)$ and $Q(A_+)$ by arc families $Q_i(A_-)$ and $Q_i(A_+)$ which are likewise as close to $Q(A)$ as we please (but are disjoint since they are near to sets that are a finite distance apart). We then examine the geometry in $\mathbf{S}^2$ of a single one of the continua $Q(A, j)$ and its two corresponding approximating arcs $Q_i(A_-, j)$ and $Q_i(A_+, j)$. All three join the centers p_1 and p_2 of the same two disk preimages B_1 and B_2 of the disk B, all three intersect B_1 and B_2 in radii, and the three intersect each other only in the two points p_1 and p_2. (See Figure 5.)

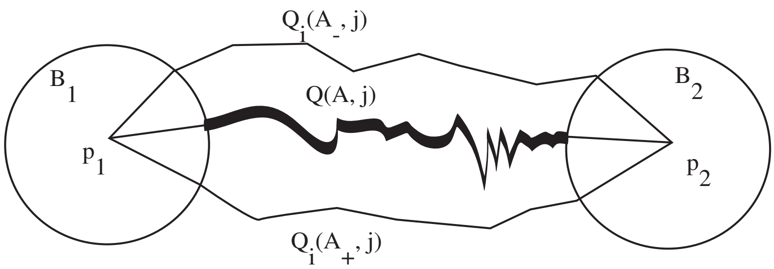


Figure 5.

The arcs $Q_i(A_-, j)$ and $Q_i(A_+, j)$ bound a disk D_j in $\mathbf{S}^2$ which contains $Q(A, j) \setminus \{p_1, p_2\}$ in its interior. It is not difficult to show, by means of the separation properties in $\mathbf{S}^2$ and the fact that $Q(A, j)$ does not separate $\mathbf{S}^2$, that, for $Q_i(A_-, j)$ and $Q_i(A_+, j)$ sufficiently close to $Q(A, j)$, the disk D_j maps under f to a small neighborhood of A in K.

We use the disk D_j to show how to homotop $Q_i(A_-, j)$ into A within K. Since the continuum $Q(A, j)$ separates $Q_i(A_-, j)$ from $Q_i(A_+, j)$ in D_j, we may express $D \setminus Q(A, j) = X_- \cup X_+$, where X_- and X_+ are disjoint open subsets of D_j, with $Q_i(A_-, j)$ in the closure of X_- and $Q_i(A_+, j)$ in the closure of X_+. Let H be a homotopy of $Q_i(A_-, j)$ across D_j to $Q_i(A_+, j)$ constructed in such a way that it is simple rotation across B_1 and B_2. Map $X_- \cup Q(A, j)$ to K by f. Map $Q(A, j) \cup X_+$ to A so as to extend $f|Q(A, j)$; take care that, over B, this map is rotation. Let g denote the union of these two maps, so that g is a new map from D_j into K. Then the composite $g \circ H$ homotops $Q_i(A_-, j)$ into A within K in the manner required by Step 3.

The arcs $Q_i(A_-, j)$, for $j = 1, \ldots, \ell$, are disjoint. Hence, each of them may be homotoped into A separately. Since these arcs are flat in $\mathbf{S}^2$, this homotopy may be extended to all of $\mathbf{S}^2$. The resulting homotopy completes the expansion of Step 3 for Case 1.

This expansion of Steps 2 and 3 for $k = 2$ completes the proof of Lemma 3 for that case. We conclude that $\pi_2(K) = 0$ for all planar sets K.

Case 2. We now modify the proof just given in Case 1 to prove a very technical lemma that will be used for $k > 2$. The setting for the lemma is illustrated by the following diagram:

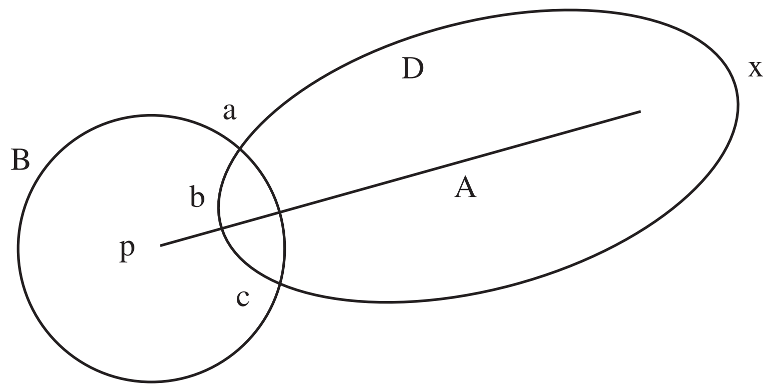


Figure 6. Setting for the Technical Lemma.

Setting:
B, a round disk in $\text{int}(K)$.
A, an arc in K from the center p of B to $\partial(K)$.
D, a disk which contains $A \setminus \text{int}(B)$ in its interior, whose boundary intersects B in a spanning arc abc of B, intersects A and is pierced by A at the single point b, and intersects $\mathbf{E}^2 \setminus K$ at a point x.
$f : \mathbf{B}^2 \to K$, a map, piecewise linear over $\text{int}(K)$, with $f(\mathbf{S}^1) \subset \text{int}(D)$, b a regular point of the map f.

Technical Lemma. There exists a disk D' arbitrarily close to D homeomorphically, finitely many disjoint simple closed curves $\{J_i\}$ in $\text{int}(\mathbf{B}^2)$, and a map $f' : \mathbf{B}^2 \to K$ homotopic to f in K such that

(i) The disks bounded by the J_i contain ${f'}^{-1}(K \setminus D)$;
(ii) For each i, $f'(J_i)$ is a single point of $\partial(D')$;
(iii) The homotopy taking f to f' fixes $f^{-1}(A)$;
(iv) The disk D', the simple closed curves $\{J_i\}$, and the homotopy may be chosen in such a way that the homotopy takes place in any prescribed neighborhood of $\partial(D)$.

Proof (Technical Lemma). The set $\partial(D')$ plays the same role in this lemma as did the arcs A in the argument for Case 1 of Lemma 3. Here too we have uncountably many disjoint choices for $\partial(D')$, with the disk D' satisfying essentially all of the same hypotheses satisfied by the original disk D relative to points $a' = a(D')$, b', c', and x', with $D' \subset \text{int}(D)$.

Step 2.1 (Case 2). *Manifolds Q_i and their limits.* For each D' and for each positive integer i, approximate $f : \mathbf{B}^2 \to K$ by a map $f_i : \mathbf{B}^2 \to \mathbf{E}^2$ which is within $1/i$ of f and agrees with f outside the $1/i$ neighborhood of $f^{-1}(\partial(D'))$. Require that f_i be transverse to $\partial(D')$ and that $f_i = f$ near $A \cap \partial(D')$. Then $f_i^{-1}(\partial(D'))$ is a finite collection of disjoint simple closed curves J_α in $\text{int}(\mathbf{B}^2)$. Let D_α be the disk in $\mathbf{B}^2$ bounded by J_α. Discard J_α if there is a β such that $D_\alpha \subset \text{int}(D_\beta)$, and discard J_α if $f_i(D_\alpha) \subset D$. We denote the remaining finite set of simple closed curves by $Q_i(D')$.

We say that two of the collections $Q_i(D')$ and $Q_j(D')$ define the same partition if they have the same number ℓ of simple closed curves and those curves can be ordered,

$$Q_i(D') = \{J_{i1}, \ldots, J_{i\ell}\} \text{ and}$$

$$Q_j(D') = \{J_{j1}, \ldots, J_{j\ell}\},$$

in such a way that both of the following two conditions are satisfied:

(i) $x \in f^{-1}(\mathbf{E}^2 \setminus D) = f_i^{-1}(\mathbf{E}^2 \setminus D) = f_j \mathrm{int}(\mathbf{E}^2 \setminus D)$ lies in the interior of the disk $D(J_{ik})$ bounded by J_{ik} iff it lies in the interior of the disk $D(J_{jk})$ bounded by J_{jk}.

(ii) The points of $f^{-1}(A \cap \partial(D')) = f_i^{-1}(A \cap \partial(D')) = f_j^{-1}(A \cap \partial(D'))$ which appear on J_{ik} appear on J_{jk} as well, and in the same cyclic order. Moreover, if the arc α bounded by two such consecutive points of of $f^{-1}(A \cap \partial(D'))$ in J_{ik} maps to one particular side of A in D, then the corresponding arc β in J_{jk} maps to the same side of A in D.

It is a simple exercise using the local compactness and local connectivity of $\mathbf{S}^2$ to prove that, for fixed D', there are only finitely many possible partitions for approximations that are sufficiently close to f. By passing to a subsequence, we may assume that, for all i and j, f_i and f_j define the same partition.

As in Case 1, we may assume that we have convergence of continuum families,

$$Q_i(D') \to Q(D').$$

Step 2.2 (Case 2). *Varying the disk D'.* We now think of the disk D' as a parameter which can be varied.

By shrinking some D' "radially," we find that we may consider an entire closed and uncountable interval of disks D'. By passing to successive subfamilies of these disks, we may restrict the properties of the associated limit continua, $Q(D') = \{J_1(D'), \ldots, J_\ell(D')\}$ in three ways, as follows:

(1) Since all of our disks D' have boundaries uniformly distant from $\partial(D)$, there are only finitely many possible partitions. Hence we may assume all share the same partition.

(2) Since, for fixed j, each $J_{ij}(D')$ separates the same two points of of $\mathbf{B}^2$, we may assume that $J_j(D') = \lim_{i\to\infty} J_{ij}(D')$ does also. Since only countably many continua of a disjoint collection can have more than two complementary domains, we may assume that each $J_j(D')$, for each D' has exactly two complementary domains.

(3) The disks D' are linearly ordered by inclusion. Hence, in the same way as in Case 1, we may assume that each $Q(D') = \{J_1(D'), \ldots, J_\ell(D')\}$ is an uncountable limit of other families $Q(D'')$ of continua from within and without.

Step 3 (Case 2). Having restricted our family of disks D' in the three ways just described, any of the (uncountably many) remaining disks D' is suitable for our purposes. For such a D', since $Q(D')$ is a limit point from both sides, we may choose D'_- and D'_+ such that $D'_- \subset \mathrm{int}(D')$ and $D' \subset \mathrm{int}(D'_+)$ and so that $Q(D'_-)$ and $Q(D'_+)$ are arbitrarily close to $Q(D')$ in the Hausdorff topology. We approximate $Q(D'_-)$ and $Q(D'_+)$ by simple closed curve families $Q_i(D'_-)$ and $Q_i(D'_+)$ which are likewise as close to $Q(D')$ as we please. We then examine the geometry in $\mathbf{S}^2$ of a single one of the continua $Q(D', j)$ and the corresponding simple closed curves $Q_i(D'_-, j)$ and $Q_i(D'_+, j)$. We shall treat the simple closed curves differently depending on whether or not their images intersect the arc A or not.

If the point b', which is the intersection of A and $\partial(D')$, lies in the image of $Q(D', j)$, then we proceed as follows. Near b', where f is already transverse and need not be altered in constructing $Q_i(D'_-)$, $Q_i(D')$, and $Q_i(D'_+)$, we see that $Q_i(D'_-, j)$ and $Q_i(D'_+, j)$ lie on opposite sides of $Q_i(D', j)$ (see Figure 6):

[The symbols $b'_1, b'_2, \ldots$ denote the (even number of) points in which $Q_i(D', j)$ crosses A. The simple closed curves $Q_i(D'_-, j)$ and $Q(D'_+, j)$ cross A at corresponding points as indicated.]

Thus the simple closed curves $Q(D'_-, j)$ and $Q(D'_+, j)$ bound an annulus R which is separated into open sets X_- and X_+ which contain $Q_i(D'_-, j)$ and $Q_i(D'_+, j)$, respectively. If b'_k and b'_{k+1} are successive points of $Q(D', j) \cap f^{-1}(b')$ on $Q(D', j)$, then we may form a simple closed curve $A_- \cup A_+$ about the "segment" between b'_k and b'_{k+1} of $Q(D', j)$ as indicated in the following figure (Figure 8):

The picture is exactly analogous to the picture considered in Case 1 and may be used to homotop A_- into $K \setminus A$ without moving the endpoints of A_-. We can then extend the arc A_- to a longer path B_- which reaches all of the way to b'_k and b'_{k+1}. The image of B_- after our homotopy is a loop in $\partial(D') \cap K$ based at b' and lying, except for the base point, to one side of A. We contract this loop to b'.

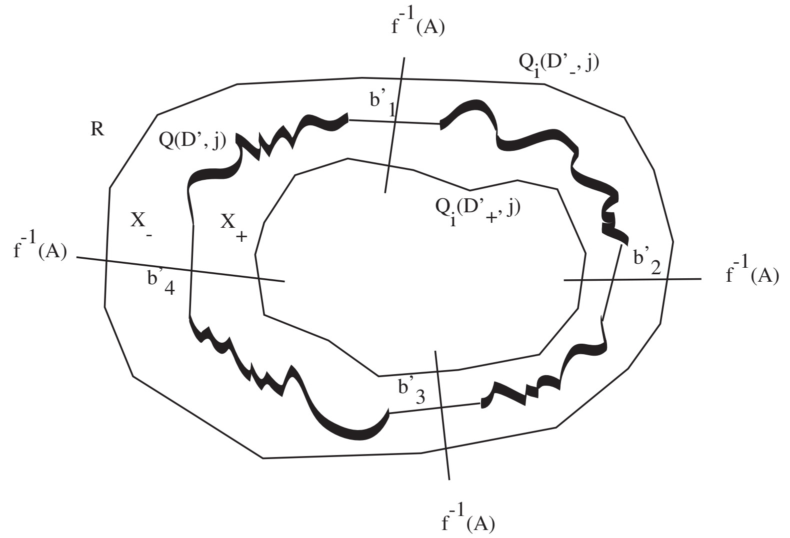


Figure 7. The case where $b' \in A \cap \partial(D')$.

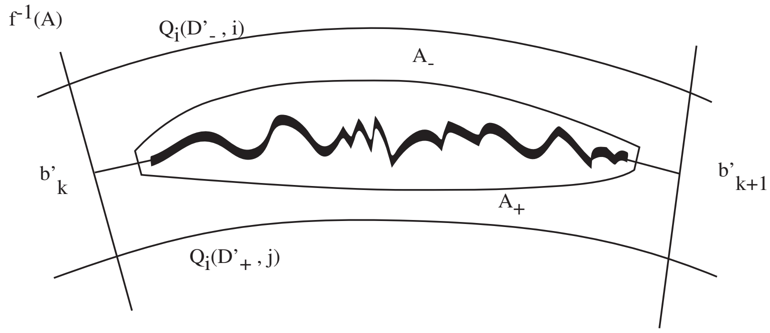


Figure 8. The simple closed curve $A_- \cup A_+$.

If the point b' does not lie in the image of $Q(D',j)$, then we cannot tell from local information that $Q_i(D'_-,j)$ and $Q_i(D'_+,j)$ bound an annulus R whose ends are separated by $Q(D',j)$, since the two simple closed curves may lie to the same side of $Q(D',j)$. They all three, however, are "concentric" in the sense that they separate a certain prescribed portion of $f^{-1}(K \setminus D)$ from $\partial(\mathbf{B}^2)$. In this case, we simply abandon the simple closed curve $Q_i(D'_+,j)$ and replace it by a simple closed curve J that does bound the desired annulus R together with $Q_i(D'_-,j)$. (See Figure 9.)

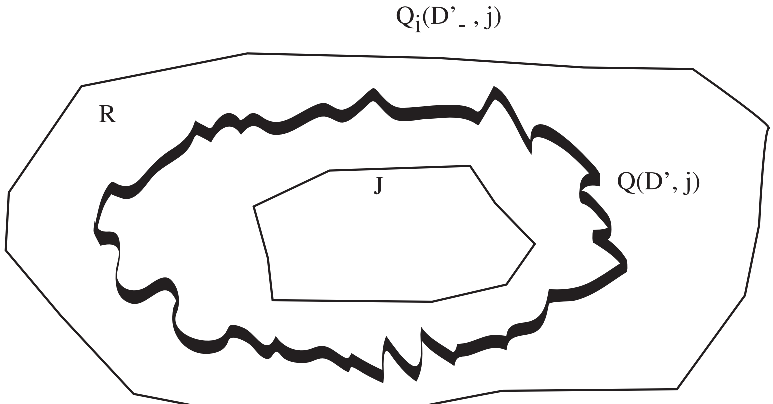


Figure 9. The annulus R.

This annulus allows us to homotop $Q_i(D'_-, j)$ into $\partial(D')$. [Since the set $Q(D', j)$ separates the ends of R, we may express $R \setminus Q(D', j) = X_- \cup X_+$, with $Q_i(D'_-, j) \subset X_-$ and $J \subset X_+$; homotop $Q_i(D'_-, j)$ across R to J; apply the Tietze extension theorem; etc.] Then the image loop can be contracted to a point of $\partial(D') \setminus A$.

Having thereby dealt with each of the possible types of continua $Q(D', j)$, we have completed the proof of the technical lemma of Case 2.

Case 3 (k > 2). By an induction which begins with Case 1, we may assume that, for any planar set L, $\pi_2(L) = \cdots = \pi_{k-1}(L) = 0$. We shall complete the proof of Lemma 3 for dimension k, and with it the fact that $\pi_k(K) = 0$, by expanding Steps 2 and 3 in this case.

Step 2.1 (Case 3). *Limits of the manifolds Q.* For a fixed arc A, we define $(k-1)$-manifolds $Q = Q_i$ as described in the discussion of Step 2 above relative to shorter and shorter extensions A' of the arc A and smaller and smaller modifications of the map f.

Each manifold Q_i defines a *partition of P* in the sense that we consider two points of P to be Q_i-equivalent if they lie in the same component of Q_i. Since there are only finitely many such partitions, we may, by passing to a subsequence, assume that all of the manifolds Q_i define the same partition of P. That is, we may assume the existence of an integer ℓ such that each Q_i has exactly ℓ components, $Q_i(1), \ldots, Q_i(\ell)$, and that, for each index a, b, and c, $\partial Q_a(c) = \partial Q_b(c)$. Since the space $C(\mathbf{S}^k)$ of compact subsets of $\mathbf{S}^k$ is a compact metric space, we may assume as in Case 1 that for $j = 1, \ldots, \ell$, the sequence $Q_i(j)$ converges to a compact subset $Q(j)$ of $f^{-1}(A)$. Since each of the sets $Q_i(j)$ is connected, it follows that $Q(j)$ is also connected.

Step 2.2 (Case 3). *Varying the arc A.* We now think of the arc A as a parameter which can be varied, and we insert this parameter into all of our Q notation.

By rotating A about p, we find that we may consider an entire interval of arcs A. By passing to successive subfamilies of these arcs, we may restrict the properties of the associated limit continua $Q_1(A), \ldots, Q_\ell(A)$ in two ways, as follows:

(1) There are only finitely many possible partitions. Hence we may assume all share the same partition of the $(k-2)$-manifold P.

(2) Each continuum family $Q(A)$ is an uncountable limit point of $Q(A')$'s both from the left and from the right in the sense described in Case 1. The proof is exactly that of Case 1 with the single exception that $k = 2$ must be replaced by general k.

Step 2.3 (Case 3). *Final choices and setup for Step 3.* We have established the existence of an uncountable family of arcs A in the neighborhood N, all defining the same partition of P, all having families $Q(A) = \{Q(A, 1), \ldots, Q(A, \ell)\}$ of limit continua with $Q(A)$ an uncountable limit of such from both left and right. We pick two of these arcs, which we call A and A'.

Let N' denote an open subset of $\mathbf{E}^2$ whose intersection with K is equal to N. Around $A \setminus \mathrm{int}(B)$ in $N' \setminus A'$ we choose a disk D exactly as in the setting for the technical lemma of Case 2. (Recall Figure 6.)

Now we choose two more of our acceptable arcs A, one to the left of A, which we label A_-; the other to the right of A, which which we label A_+. Since $Q(A)$ is an uncountable limit from both right and left, we may choose A_- and A_+ so that the Hausdorff distances among $Q(A_-)$, $Q(A)$, and $Q(A_+)$ are all so small that every point of each which does not map into B under f can be joined to a point of the other by an arc which maps into $\mathrm{int}(D)$. Finally, we approximate $Q(A_-)$ and $Q(A_+)$ by the $(k-1)$-manifolds $Q_i(A_-)$ and $Q_i(A_+)$ with i so large that our restrictions on Hausdorff distance still apply. That is, every point of each which does not map into B under f can be joined to a point of the other by an arc which maps into $\mathrm{int}(D)$.

We are now ready for the expanded version of Step 3.

Step 3 (Case 3). Our goal is to homotop the image under f of the $(k-1)$-manifold $Q_i(A_-)$ into the arc A in K by a homotopy which moves points only in N.

The homotopy is already prescribed near B to be a simple rotation. In order that we not have to worry further about this rotation near B, we can simply collapse the sector or angle of radii between A_- and A_+ to a single radius, and the inverse of this sector under f in the same fashion, so as to move $Q_i(A_-) \cap f^{-1}(B)$ to coincide with $Q_i(A_+) \cap f^{-1}(B)$. See Figure 11.

At the same time, we may adjust the disk D so that its intersection with the radius to which the sector

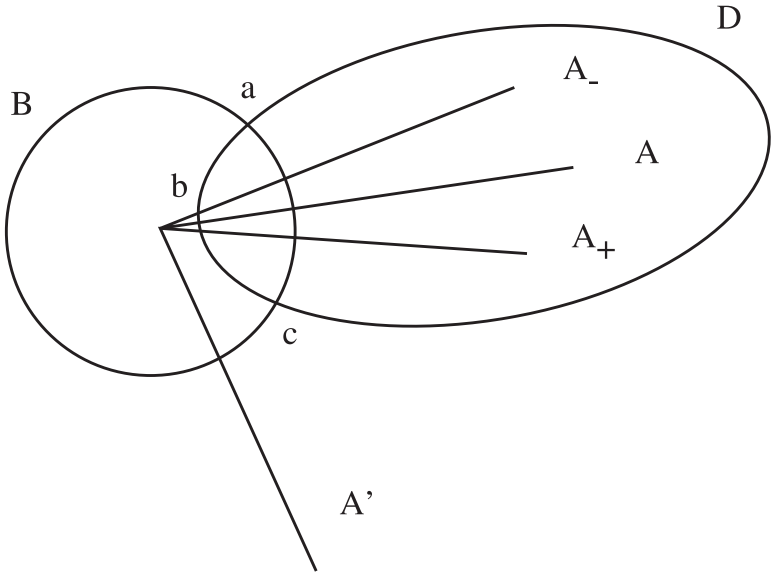


Figure 10.

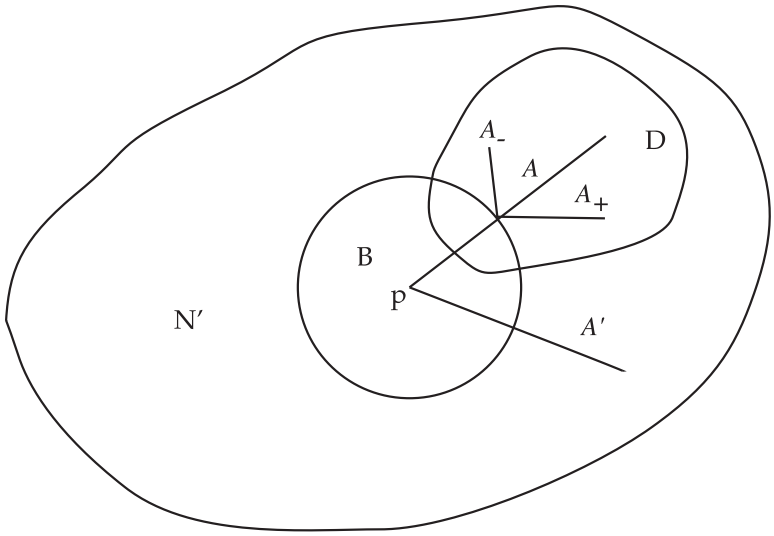


Figure 11. Collapsing the sector.

of radii has been collapsed is a regular point and that, moreover f is transverse over the arc $\partial(D) \cap B$.

We may assume that $Q_i(A_-)$ is triangulated and covered by a complex L, with $Q_i(A_-) \cap f^{-1}(B)$ being covered by a subcomplex. We may assume that the simplexes σ of L are so small that if the image of σ hits B, then the entire image of σ is very near B. We shall homotop L skeleton by skeleton. We first show that the entire problem can be easily reduced to constructing the homotopy of the 1-skeleton of L.

Step 3.1 (Case 3). *Every homotopy of the 1-skeleton $L^{(1)}$ into A can be completed to a homotopy of L into A.* Let $H : L^{(1)} \times [0,1] \to K$ be a continuous function such that

(1) $H(x,0) = f(x), \forall x \in L^{(1)}$.

(2) $H(x,1) \in A, \forall x \in L^{(1)}$.

(3) Over B, $H(x,t) = f(x), \forall x \in L^{(1)}$.

(4) Away from B, H moves points only in the disk D.

Assume inductively that H is defined on some $L^{(i)}$, with $i \geq 1$, satisfying the analogues of (1), (2), (3), and (4). Suppose that σ is an $(i+1)$-simplex in L; note that $(i+1) \leq (k-1)$. Then $H(\partial(\sigma) \times \{1\})$ is a singular i-sphere in A. If σ has image in B, then, using (3), the homotopy need not move σ. Otherwise, using (4), the image lies in $A \cap D$, and we can define $H \mid \sigma \times \{1\}$ to take $\sigma \times \{1\}$ into the very image of $H \mid \partial(\sigma) \times \{1\}$. In the latter case, the map H is then defined on the $(i+1)$-sphere $(\sigma \times \{0\}) \cup (\partial\sigma \times [0,1]) \cup (\sigma \times \{1\})$. Since $(i+1) \leq (k-1) < k$, we may assume by induction on k that the latter sphere may be contracted in its own image, which lies in D. This completes the definition of H on $\sigma \times [0,1]$.

Before we can homotop the 1-skeleton of $L^{(1)}$ into A, we need the following lemma.

Step 3.2 (Case 3). *Separation Lemma.*

Let M be one of the continua comprising the continuum family $Q(A)$. Let M_- denote the corresponding component of the $(k-1)$-manifold $Q_i(A_-)$ and M_+ the corresponding component of the $(k-1)$-manifold $Q_i(A_+)$.

Separation Lemma. If γ is any arc in $\mathbf{S}^k$ which joins M_- and M_+, and if $f(\gamma)$ is a subset of $\mathrm{int}(D)$, then γ intersects M.

Proof. Since M_- and M_+ agree over B, it is an easy matter to form a loop $\delta : \mathbf{S}^1 \to \mathbf{S}^k$ which is the concatenation of the path γ and a path γ', where γ' is the concatenation of two paths, namely a path in M_+ which begins at the end of γ in M_+ and ends at a point which projects under f to $\partial(B)$ and a path in M_- outward to the endpoint of γ in M_-. Note that the loop $\delta = \gamma \cdot \gamma'$ projects under f into $\mathrm{int}(D)$. See Figure 12.

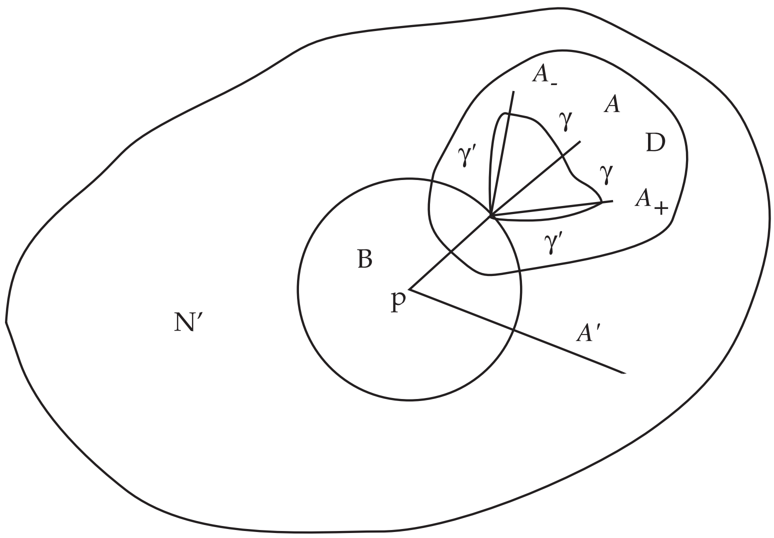


Figure 12. The projection of the loop $\delta = \gamma\gamma'$.

Since $\mathbf{S}^k$ is simply connected, the loop δ bounds a singular disk $\epsilon : \mathbf{B}^2 \to \mathbf{S}^k$. We may assume that the singular disk ϵ is in general position with respect to $\partial(M_-) = \partial(M_+)$.

We claim the preimage of $\partial(M_-)$ in the domain of the singular disk ϵ consists of an even number of points. Indeed, recall the arc $A' \subset K$ which we chose in the first paragraph of Step 2.3 which is disjoint from D. One of the constituent continua M' of $Q(A')$ corresponds to to the continuum M in the sense that its "boundary" is equal to the "boundary" of M. This continuum is approximated arbitrarily closely by $(k-1)$-manifolds M'' in $\mathbf{S}^k$ having the same boundary; in fact, it was defined originally as a limit of such manifolds. Note that we may pick the approximation M'' in such a way that its image under f entirely misses D since it is a close approximation to a set whose image lies in A'. Hence, we may put M'' in general position with respect to ϵ in such a way that it misses the boundary loop δ. The inverse image of M'' in $\mathbf{B}^2 =$ (domain of ϵ) will consist of simple closed curves, which we ignore, and a finite number of arcs beginning and ending at distinct points of the preimage of $\partial(M_-)$. Hence there are an even number of such points.

In exactly the same way, it is possible to approximate $M \subset f^{-1}(A)$, instead of $M' \subset f^{-1}(A')$, by $(k-1)$-manifolds M'', all having the same boundary $\subset P$. But this new M'' will have image in $B \cup D$. Put this new M'' in general position with respect to ϵ. The inverse of M'' in $\mathbf{B}^2$ will again consist of simple closed curves, which we shall ignore, and of a finite number of arcs beginning and ending at the even number of points considered in the previous paragraph, at exactly one point of the domain of γ', and perhaps at some point or points of the domain of γ. But the total number of end points is even; hence there must be such a point on the domain of γ. Now letting the M'' vary and converge to M, we find that M too must intersect the image of γ. This completes the proof of the lemma.

Step 3.3 (Case 3). *Homotoping the 1-skeleton $L^{(1)}$ into A: preparations.* Recall that L is a complex covering the $(k-1)$-manifold $Q_i(A_-)$.

Let M be one of the continua comprising the continuum family $Q(A)$. Let M_- denote the corresponding component of the $(k-1)$-manifold $Q_i(A_-)$ and M_+ the corresponding component of the $(k-1)$-manifold $Q_i(A_+)$. We concentrate on homotoping the 1-skeleton $M_-^{(1)}$ of the subcomplex of L covered by M_-.

Our goal is to construct a map $H : M_-^{(1)} \times [0,1] \to K$ so as to satisfy (1), (2), (3), and (4) of Step 3.1. Near B we have no choices to make because of (3). Away from B we proceed as follows.

Step 3.4 (Case 3). *The homotopy $G : M_-^{(1)} \times [0,1] \to \mathbf{S}^k$.* First we map $M_-^{(1)} \times [0,1]$ into $\mathbf{S}^k$ by an auxiliary map G. The initial end $M_-^{(1)} \times \{0\}$ maps by the identity.

If v is a vertex of $M_-^{(1)}$, then we define $G(v,1)$ to be a point of M_+ that is closest to v, and we define $G \mid \{v\} \times [0,1]$ to be a geodesic in $\mathbf{S}^k$ joining v and $G(v,1)$. This geodesic will lie very close to M because the manifolds M_- and M_+ are Hausdorff approximations to M.

If σ is an edge (1-simplex) of $M_-^{(1)}$ with endpoints v and w, then because $G(v,1)$ and $G(w,1)$ lie in the same component M_+ of $Q_i(A_+)$, there is an arc $G(\sigma \times \{1\})$ in M_+ joining those two points. Since M_+ is so simple over B, we may require that this arc miss $f^{-1}(\mathrm{int}(B))$, hence lie in $f^{-1}(\mathrm{int}(D))$.

For each edge σ as in the previous paragraph, the map G is defined on the (square) simple closed curve $J(\sigma) = (\sigma \times \{0,1\}) \cup (\partial(\sigma) \times [0,1])$, and the image of the square curve $J(\sigma)$ lies in the f-preimage of $\mathrm{int}(D)$. Since $\mathbf{S}^k$ is simply connected, we may extend G to the disk $D(\sigma)$ in $M_-^{(1)} \times [0,1]$ spanned by $J(\sigma)$.

At this point, we have defined a homotopy $G : M_-^{(1)} \times [0,1] \to \mathbf{S}^k$.

Step 3.5 (Case 3). *Finishing the proof: the simple case.* If the image of $f \circ G$ were to lie in N, then we could complete the proof as follows. By the Separation Lemma, $M_-^{(1)} \times [0,1] \setminus G^{-1}(M) = X \cup Y$, where X and Y are disjoint open subsets of $M_-^{(1)} \times [0,1]$, X containing $M_-^{(1)} \times \{0\}$ and Y containing $M_-^{(1)} \times \{1\}$. By the Tietze extension theorem, there is a map $h : Y \cup G^{-1}(M) \to A$ which extends $f \circ G \mid G^{-1}(M)$. Define $h \mid X \cup G^{-1}(M) = f \circ G \mid X \cup G^{-1}(M)$. By the pasting lemma, h is a continuous function. It homotops $M_-^{(1)}$ into A as desired.

Step 3.6 (Case 3). *Finishing the proof.*

Unfortunately, the singular disks $f(G(D(\sigma)))$ will probably stick out of D in K. Only finitely many components of $D(\sigma) \setminus G^{-1} \circ f^{-1}(\partial D)$ have images which protrude substantially outside of D. These components we need to truncate in a nice way. Actually, by considering a smaller disk D' just inside of D, we may truncate all the components that hit $\partial(D)$.

We have precisely the setting of the Technical Lemma proved in Case 2. Hence we may homotop $f \circ G$ without moving $G^{-1}(M)$ so that the components which protrude beyond D' enough to hit $\partial(D)$ satisfy the conditions: (i) these components are bounded by simple closed curves in $\mathrm{int}(D(\sigma))$ and (ii) those simple closed curves are mapped by f to single points of $K \cap \partial(D')$. Hence we are prepared to complete the construction of the homotopy of the 1-skeleton as follows.

Let $\{J_i\}$ denote the collection of simple closed curves in $M_-^{(1)} \times [0,1]$ described in the previous paragraph, let M_0 denote the space formed from $M_-^{(1)} \times [0,1]$ by deleting the interiors of the simple closed curves J_i. There is an identification map $\pi : M_0 \to M_-^{(1)} \times [0,1]$ which identifies each J_i to a single point. By (ii) above, the map $f \circ G \mid M_0$ factors through the projection π. By the Separation Lemma, the set $G^{-1}(M)$ separates $M_-^{(1)} \times \{0\}$ from $M_-^{(1)} \times \{1\}$ in M_0. Hence $\pi(G^{-1}(M))$ separates these two sets in $M_-^{(1)} \times [0,1]$. Because the homotopy of $f \circ G$ used in satisfying (i) and (ii) did not disturb $G^{-1}(M)$, we may use the separation $M_-^{(1)} \times [0,1] \setminus \pi(G^{-1}(M)) = X \cup Y$, with $M_-^{(1)} \times \{0\} \subset X$ and $M_-^{(1)} \times \{1\} \subset Y$, exactly as above to homotop $M_-^{(1)}$ into A. The required map h, on $X \cup \pi(G^{-1}(M))$, is simply $f \circ G \circ \pi^{-1}$. On $Y \cup \pi(G^{-1}(M))$ one uses the Tietze extension theorem as before.

The proof of Lemma 3 is finally complete, and, with it, the proof of the main theorem.

5. A planar Peano continuum that is not homotopy equivalent to a 1-dimensional space.

5.1. Zastrow's example. Let S denote the standard Sierpinski curve in $\mathbf{E}^2$. Let X be the space formed by filling one of the bounded complementary domains of S. See Figure 13.

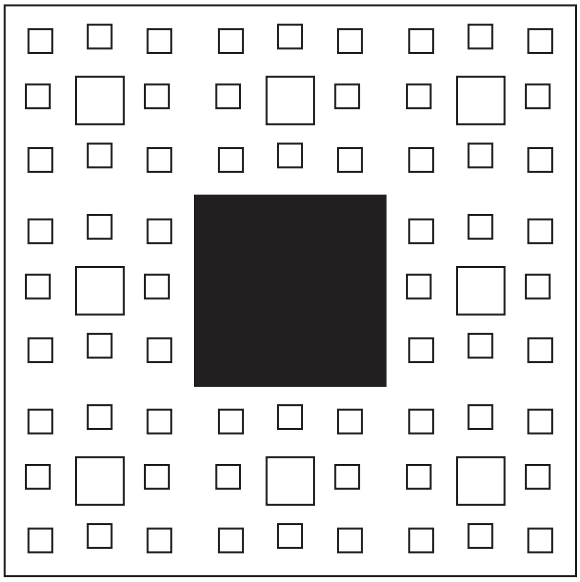


Figure 13. Sierpinski curve with central disk filled.

5.2. Theorem. Zastrow's filled Sierpinski curve X is not homotopy equivalent to a 1-dimensional metric space.

Proof. Suppose there were a 1-dimensional metric space Y and inverse homotopy equivalences $f : X \to Y$ and $g : Y \to X$. Let J be the simple closed curve that is the boundary of the filled complementary domain in going from the Sierpinski curve S to the space X. There are then two steps to our proof.

(1) The map $f \,|\, J$ is not 1-1.

(2) The map $g \circ f \,|\, J$ is the identity.

The incompatibility of (1) and (2) shows that our assumption about the existence of Y, f, and g was incorrect, so that the theorem is true.

If the map $f \,|\, J$ were 1-1, then it would define a reduced loop in a 1-dimensional space so that, by Theorem 3.9 of [CC], $f \,|\, J$ could not be nullhomotopic in Y, a clear absurdity. Thus (1) is established.

Let $x \in J$. Suppose that $g \circ f(x) = x' \neq x$. Let D be a disk in $\mathbf{E}^2$ which contains x in its interior and x' in its exterior, and whose boundary intersects X in an arc α. Let K be the boundary of a complementary domain of X in $\mathbf{E}^2$ that is so close to x that it lies in the interior of D and its image $g \circ f(K)$ lies in the exterior of D. Since f and g are inverse homotopy equivalences, the map $g \circ f$ is homotopic to the identity. Hence $\mathrm{id} \,|\, K$ and $g \circ f \,|\, K$ are freely homotopic in X. The homotopy between them may be cut off on $\alpha = \partial(D) \cap X$ by Tietze's extension theorem. Since α is contractible, the annulus which homotops $\mathrm{id} \,|\, K$ into α can be completed to a contraction in X. That is, K is nullhomotopic in X, an absurdity. We conclude that (2) is true.

The proof that X is not homotopy equivalent to a 1-dimensional metric space is therefore complete.

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