Group Action Systems:
a Mathematical tool for deriving Provable Secure Cryptographic Schemes

María Isabel González Vasco

Universidad Rey Juan Carlos

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Joint works with J. L. Villar (UPC) and R. Steinwandt (FAU)

Overview

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  - Definitions
  - Basic Results
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- Final Remarks

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- Fact: work in that direction hardly exploits the constructions and theoretical frameworks available from number-theoretical cryptography.
- Our Goal: adapt the existing theory of Universal Projective Hash Functions to allow constructions arising in different areas of mathematics .

Some basics about PHFs

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Let X, Π, S be non-empty sets, L⊆ X, and K a finite index set. Consider $H := \{ H_k : X \mapsto \Pi \}_{k \in K}$ and $\alpha : K \mapsto S$.

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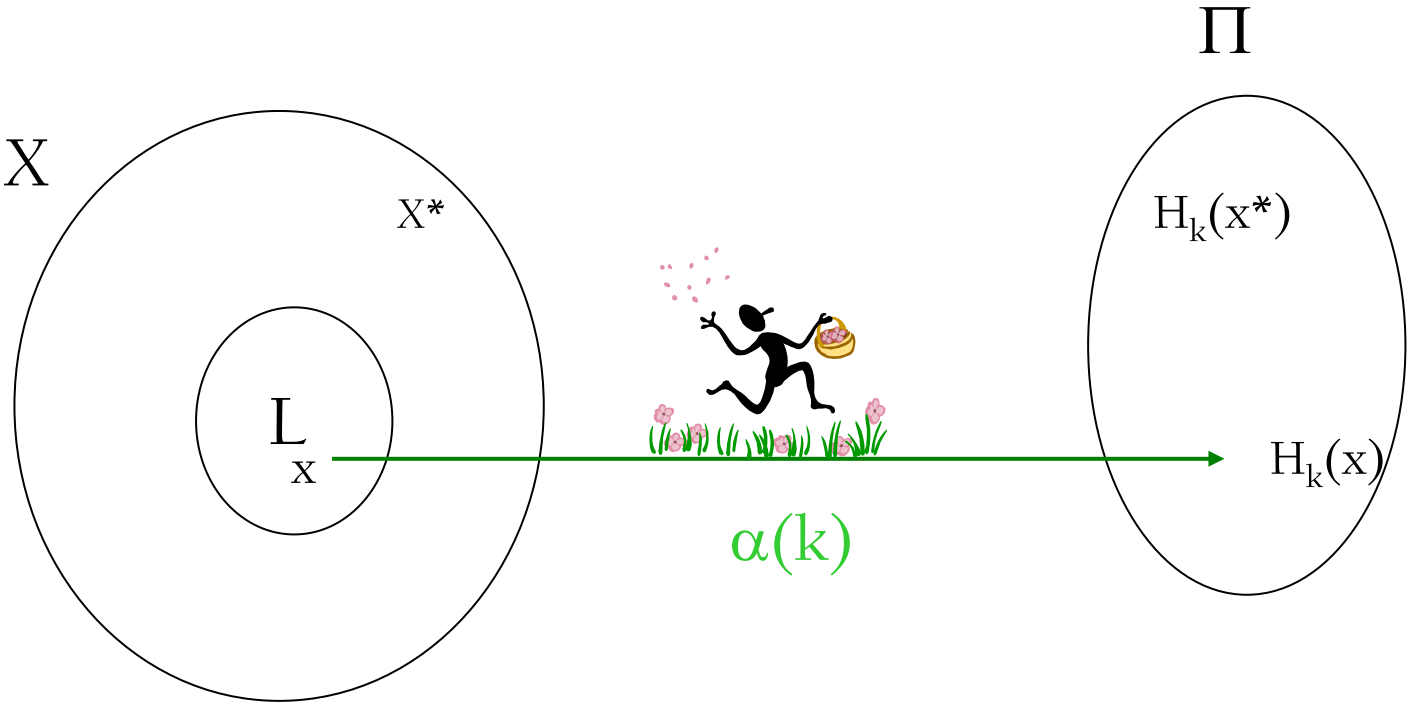
Then the tuple **H** = (H, K, X, L, Π, S, α) is a *projective hash family* - PHF - for (X, L) provided that

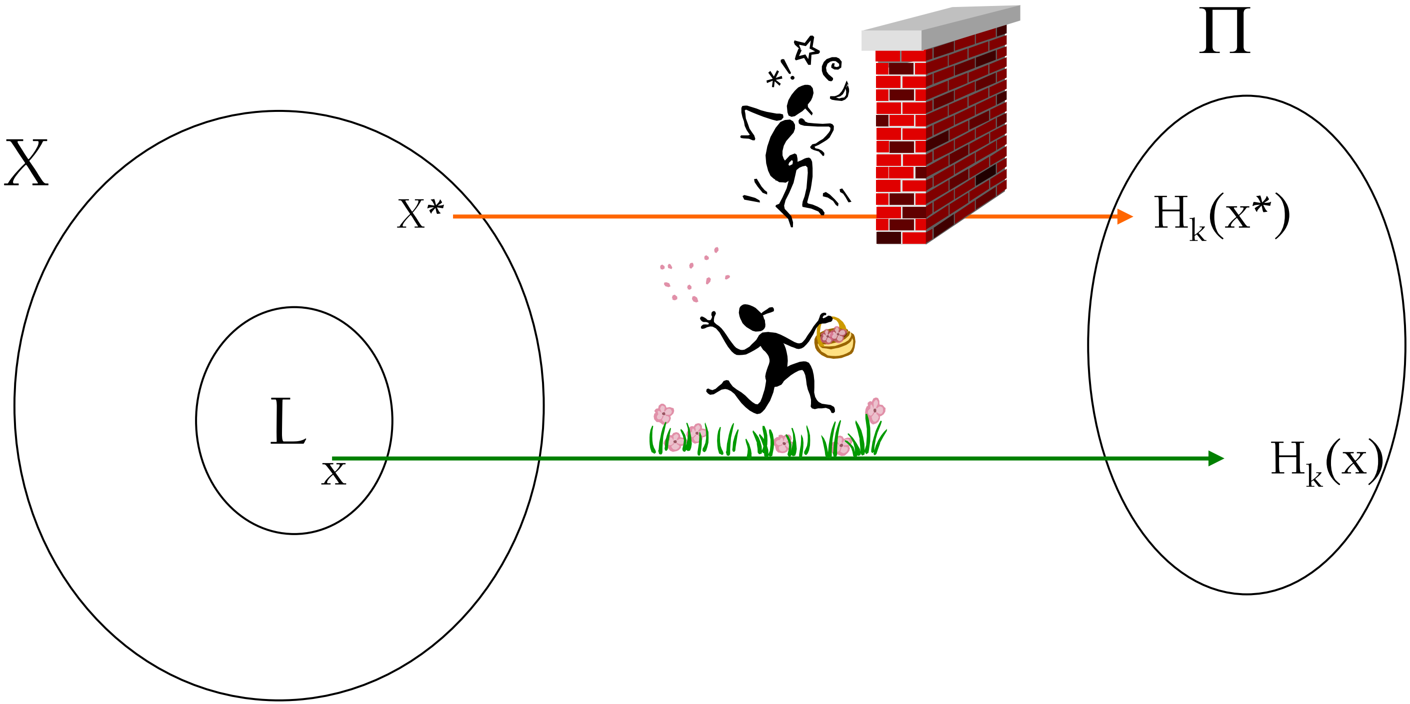
$$\alpha(k) \approx H_{k|L}()$$

(i.e., $\forall\, x \in L, k_1, k_2 \in K,\ \alpha(k_1) = \alpha(k_2) \Rightarrow H_{k_1}(x) = H_{k_2}(x)$).

Given only the projection $\alpha(k)$...



…it could be hard to compute H_k outside L



Definitions

Moreover, we say that **H** = (H, K, X, L , Π, S, α) is

- ε-universal :⇔ $\forall s \in S, x \in X \backslash L, \pi \in \Pi$

  $$P[H_k(x) = \pi \,/\, \alpha(k) = s] \leq \varepsilon ;$$

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- ε- smooth $:\Leftrightarrow$ $(x, \alpha(k), H_k(x))$ and $(x, \alpha(k), \pi)$ are ε-close for $k \in K$, $x \in X\setminus L$ and $\pi \in \Pi$ chosen uniformly at random;

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- Strongly universal$_2$ ≈ worst case smoothness.

Basic Results

- Ways of "upgrading" the weaker types of PHFs to achieve more robust types:
  - Universal to universal$_2$ - Cramer and Shoup, [EUROCRYPT 2002]
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- Methods for constructing cryptographically useful PHFs

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- Kalai [EUROCRYPT 2005]
  - 2-out-of-1 oblivious transfer protocol.

Cryptographic Applications

- Cramer and Shoup [EUROCRYPT 2002]
  - Π is the message space
  - k is kept secret, $\alpha(k)$ and x are public
  - $m \in \Pi$ is encrypted using $H_k(x)$ as a one time pad, for $x \in L$, i.e.,

$$E(\alpha(k))(m) = (x, H_k(x) \oplus m)$$

  - IND-CCA security is achieved by appending a proof of integrity

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Sender's (B) input: two strings γ_0, γ_1,

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Goal: A learns γ_b, but nothing about γ_{b-1} . B learns nothing about b.

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  - B chooses independently two random keys k_0, k_1 and sends $\alpha(k_0)$, $\alpha(k_1)$, $y_0 = \gamma_0 \oplus H_{k_0}(x_0)$ and $y_1 = \gamma_1 \oplus H_{k_1}(x_1)$;

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- A retrieves γ_b by computing $y_b \oplus H_{k_b}(x_b)$ using the projection key $\alpha(k_b)$. Note that as $x_{1-b} \in X \backslash L$, $\alpha(k_{1-b})$ does not give enough information for computing $H_{k_{1-b}}$ outside L.

Group Action Based Projective Hash Families

Group Systems

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- To derive a PHF, one must specify the action of H on L in terms of a set $\{g_1,\dots,g_d\}$ of generators for L, i.e.

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- Using group systems, they derived instances of their encryption scheme based on the DDH problem and the Decision Composite Residuosity assumption.

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Let S be a finite group and $\chi: H \mapsto S$ a group homorphism.

Then, the tuple (X, H, χ, S) is called a *group action system.*

Group Action Systems (II)

Given a group action system (X, H, χ, S), a PHF can be constructed via a suitable indexing of H, i.e., given a finite set K, $\hbar : K \mapsto H$ the tuple

$(X, H, K, S, \chi, \hbar)$ defines a PHF (AcPHF)

$$\mathbf{H} = (H, K, X, L, X, S, \chi \circ \hbar),$$

where

$$L := \{ x \in X \mid |(\mathrm{Ker}\chi)(x)| = 1 \}.$$

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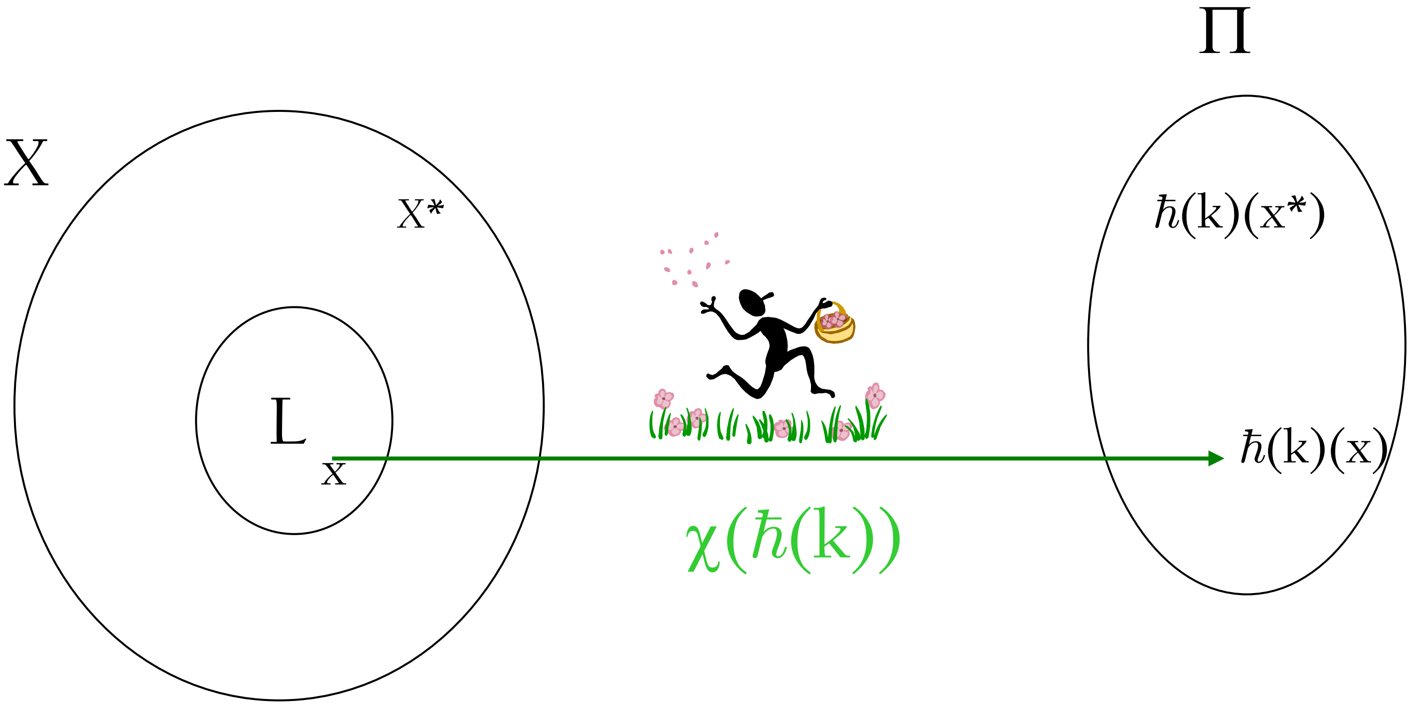
Note that:

- $L := \{ x \in X \mid (\mathrm{Ker}\chi)(x) = x \}$;
- $\mathrm{Ker}\chi \subseteq \mathrm{Stab}(L)$;
- H leaves L invariant;
- We will be interested in systems for which the ($\mathrm{Ker}\chi$)–orbits of elements in $X \backslash L$ are large.

AcPHFs



Useful AcPHFs.

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Moreover…

…there´s a "dedicated" way of upgrading it to $(1/p)$-universal$_2$!!

Examples

An example using linear groups

Let X be F_q^n, $\{\alpha_1,\ldots, \alpha_n\}$ and F_q basis for X.

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...How to achieve p-diversity?

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Construct a projection $\chi: H \mapsto H_{|L}$ by means of a "group base" of L; i.e., a sequence $[\alpha_1, \dots, \alpha_n]$, with each $\alpha_i = (\alpha_{i1}, \dots, \alpha_{ir_i})$, $\alpha_{ij_i} \in G$, so that each $g \in L$ can be expressed as a product:

$$g = \alpha_{1j_1} \cdots \alpha_{sj_s}, \text{ where } \alpha_{ij_i} \in \alpha_i .$$

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$$g = \alpha_{1j_1} \cdots \alpha_{sj_s}, \text{ where } \alpha_{ij_i} \in \alpha_i .$$

Then,

$$\begin{aligned} \chi : H &\mapsto H_{|L} \\ h &\mapsto (h(\alpha_{1j_1}),\dots,h(\alpha_{sj_s})) \end{aligned}$$

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An example using non-abelian groups (II)

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For instance, for realising Cramer and Shoup´s scheme:

- random elements from L must be hard to distinguish from random elements from X.
- "factoring" $x \in L$ with respect to the group base α should be hard (without trapdoor information)

(for details, see G-V, Martínez, Steinwandt, Villar [TCC 05])

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Define χ as the group homomorphism

$$\chi : H \mapsto S_L$$
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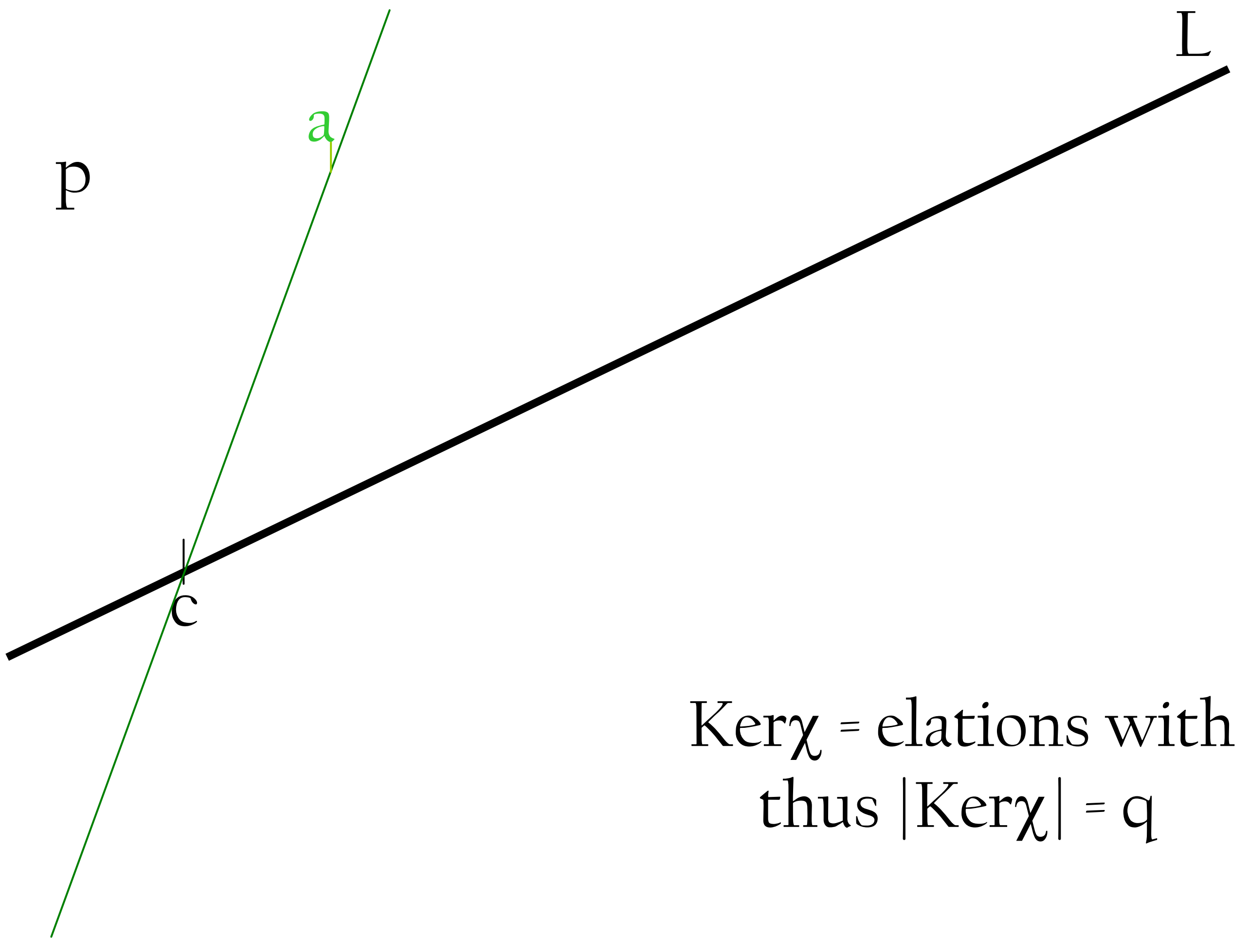

Kerχ = elations with axis L,
thus |Kerχ| = q

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- Unfortunately, so far “good” ≠ “good enough”, as the main cryptographic constructions require aditional properties.
- However, this framework sheds some light on how to use (robust enough) problems not yet exploited.

Thank you!!!