

Solutions of Selected Problems

October 21, 2010

Chapter I

1.9 Consider the potential equation in the disk $\Omega := \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$, with the boundary condition

$$\frac{\partial}{\partial r} u(x) = g(x) \quad \text{for } x \in \partial\Omega$$

on the derivative in the normal direction. Find the solution when g is given by the Fourier series

$$g(\cos \phi, \sin \phi) = \sum_{k=1}^{\infty} (a_k \cos k\phi + b_k \sin k\phi)$$

without a constant term. (The reason for the lack of a constant term will be explained in Ch. II, §3.)

Solution. Consider the function

$$u(r, \phi) := \sum_{k=1}^{\infty} \frac{r^k}{k} (a_k \cos k\phi + b_k \sin k\phi). \quad (1.20)$$

Since the partial derivatives $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \phi}$ refer to orthogonal directions (on the unit circle), we obtain $\frac{\partial}{\partial r} u$ by evaluating the derivative of (1.20). The values for $r = 1$ show that we have a solution. Note that the solution is unique only up to a constant. \square

1.12 Suppose u is a solution of the wave equation, and that at time $t = 0$, u is zero outside of a bounded set. Show that the energy

$$\int_{\mathbb{R}^d} [u_t^2 + c^2(\text{grad } u)^2] dx \quad (1.19)$$

is constant.

Hint: Write the wave equation in the symmetric form

$$\begin{aligned} u_t &= c \operatorname{div} v, \\ v_t &= c \operatorname{grad} u, \end{aligned}$$

and represent the time derivative of the integrand in (1.19) as the divergence of an appropriate expression.

Solution. We take the derivative of the integrand and use the differential equations

$$\begin{aligned}
 & \frac{\partial}{\partial t} \int_{\mathbb{R}^d} [u_t^2 + c^2(\text{grad } u)^2] dx \\
 &= \frac{\partial}{\partial t} \int_{\mathbb{R}^d} c^2[(\text{div } v)^2 + (\text{grad } u)^2] dx \\
 &= c^2 \int_{\mathbb{R}^d} [2 \text{div } v \text{div } \frac{\partial}{\partial t} v + 2 \text{grad } u \text{grad } \frac{\partial}{\partial t} u] dx \\
 &= 2c^3 \int_{\mathbb{R}^d} [\text{div } v \text{div } \text{grad } u + \text{grad } u \text{grad } \text{div } v] dx \\
 &= 2c^3 \int_{\mathbb{R}^d} \text{div}[\text{div } v \text{grad } u] dx.
 \end{aligned}$$

The integrand vanishes outside the interior of a bounded set Ω . By Gauss' integral theorem the integral above equals

$$2c^3 \int_{\partial\Omega} \text{div } v \text{grad } u \cdot n ds = 0. \quad \square$$

Chapter II

1.10 Let Ω be a bounded domain. With the help of Friedrichs' inequality, show that the constant function $u = 1$ is not contained in $H_0^1(\Omega)$, and thus $H_0^1(\Omega)$ is a proper subspace of $H^1(\Omega)$.

Solution. If the function $u = 1$ would belong to H_0^1 , then Friedrichs' inequality would imply $\|u\|_0 \leq c\|u\|_1 = 0$. This contradicts $\|u\|_0 = \mu(\Omega)^{1/2} > 0$. \square

1.12 A variant of Friedrichs' inequality. Let Ω be a domain which satisfies the hypothesis of Theorem 1.9. Then there is a constant $c = c(\Omega)$ such that

$$\|v\|_0 \leq c(|\bar{v}| + |v|_1) \quad \text{for all } v \in H^1(\Omega) \quad (1.11)$$

$$\text{with } \bar{v} = \frac{1}{\mu(\Omega)} \int_{\Omega} v(x) dx.$$

Hint: This variant of Friedrichs' inequality can be established using the technique from the proof of the inequality 1.5 only under restrictive conditions on the domain. Use the compactness of $H^1(\Omega) \hookrightarrow L_2(\Omega)$ in the same way as in the proof of Lemma 6.2 below.

Solution. Suppose that (1.11) does not hold. Then there is a sequence $\{v_n\}$ such that

$$\|v_n\| = 1 \quad \text{and} \quad |\bar{v}_n| + |v_n|_1 \leq n \quad \text{for all } n = 1, 2, \dots$$

Since $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact, a subsequence converges in $L_2(\Omega)$. After going to a subsequence if necessary, we assume that the sequence itself converges. It is a Cauchy sequence in $L_2(\Omega)$. The triangle inequality yields $|v_n - v_m|_1 \leq |v_n|_1 + |v_m|_1$, and $\{v_n\}$ is a Cauchy sequence in $H^1(\Omega)$.

Let $u = \lim_{n \rightarrow \infty} v_n$. From $|u|_1 = \lim_{n \rightarrow \infty} |v_n|_1 = 0$ it follows that u is a constant function, and from $\bar{u} = 0$ we conclude that $u = 0$. This contradicts $\|u\|_0 = \lim_{n \rightarrow \infty} \|v_n\|_0 = 1$. \square

1.14 Exhibit a function in $C[0, 1]$ which is not contained in $H^1[0, 1]$. – To illustrate that $H_0^0(\Omega) = H^0(\Omega)$, exhibit a sequence in $C_0^\infty(0, 1)$ which converges to the constant function $v = 1$ in the $L_2[0, 1]$ sense.

Solution. Let $0 < \alpha < 1/2$. The function $v := x^\alpha$ is continuous on $[0, 1]$, but $v' = \alpha x^{\alpha-1}$ is not square integrable. Hence, $v \in C[0, 1]$ and $v' \notin H^1[0, 1]$.

Consider the sequence

$$v_n := 1 + e^{-n} - e^{-nx} - e^{-n(1-x)}, \quad n = 1, 2, 3, \dots$$

Note that the deviation of v_n from 1 is very small for $e^{-\sqrt{n}} < x < 1 - e^{-\sqrt{n}}$, and that there is the obvious uniform bound $|v_n(x)| \leq 2$ in $[0, 1]$. Therefore, $\{v_n\}$ provides a sequence as requested. \square

1.15 Let ℓ_p denote the space of infinite sequences (x_1, x_2, \dots) satisfying the condition $\sum_k |x_k|^p < \infty$. It is a Banach space with the norm

$$\|x\|_p := \|x\|_{\ell_p} := \left(\sum_k |x_k|^p \right)^{1/p}, \quad 1 \leq p < \infty.$$

Since $\|\cdot\|_2 \leq \|\cdot\|_1$, the imbedding $\ell_1 \hookrightarrow \ell_2$ is continuous. Is it also compact?

Solution. For completeness we note that $\sum_i |x_i|^2 \leq (\sum_i |x_i|)^2$, and $\|x\|_2 \leq \|x\|_1$ is indeed true.

Next consider the sequence $\{x^j\}_{j=1}^\infty$, where the j -th component of x^j equals 1 and all other components vanish. Obviously, the sequence belongs to the unit ball in ℓ_1 , but there is no subsequence that converges in ℓ_2 . The imbedding is not compact. \square

1.16 Consider(a) the Fourier series $\sum_{k=-\infty}^{+\infty} c_k e^{ikx}$ on $[0, 2\pi]$,(b) the Fourier series $\sum_{k,\ell=-\infty}^{+\infty} c_{k\ell} e^{ikx+i\ell y}$ on $[0, 2\pi]^2$.

Express the condition $u \in H^m$ in terms of the coefficients. In particular, show the equivalence of the assertions $u \in L_2$ and $c \in \ell_2$.

Show that in case (b), $u_{xx} + u_{yy} \in L^2$ implies $u_{xy} \in L^2$.

Solution. Let $v(x, y) = \sum_{k=-\infty}^{+\infty} c_k e^{ikx}$. The equivalence of $v \in L_2$ and $c \in \ell_2$ is a standard result of Fourier analysis. In particular,

$$v_x \in L_2 \Leftrightarrow \sum_{k\ell} |kc_{k\ell}|^2 < \infty,$$

$$v_y \in L_2 \Leftrightarrow \sum_{k\ell} |\ell c_{k\ell}|^2 < \infty,$$

$$v_{xx} \in L_2 \Leftrightarrow \sum_{k\ell} |k^2 c_{k\ell}|^2 < \infty,$$

$$v_{xy} \in L_2 \Leftrightarrow \sum_{k\ell} |k\ell c_{k\ell}|^2 < \infty,$$

$$v_{yy} \in L_2 \Leftrightarrow \sum_{k\ell} |\ell^2 c_{k\ell}|^2 < \infty.$$

If $v_{xx} + v_{yy} \in L_2$, then $\sum_{k\ell} |(k^2 + \ell^2)c_{k\ell}|^2 < \infty$. It follows immediately that v_{xx} and v_{yy} belong to L_2 . Young's inequality $2|k\ell| \leq k^2 + \ell^2$ yields $\sum_{k\ell} |k\ell c_{k\ell}|^2 < \infty$ and $v_{xy} \in L_2$.

A simple regularity result for the solution of the Poisson equation on $[0, \pi]^2$ is obtained from these considerations. Let $f \in L_2([0, \pi]^2)$. We extend the domain to $[-\pi, \pi]^2$ by setting

$$f(-x, y) = -f(x, y), \quad f(x, -y) = -f(x, y),$$

and have an expansion

$$f(x, y) = \sum_{k,\ell=1}^{\infty} c_{k\ell} \sin kx \sin \ell y.$$

Since all the involved sums are absolutely convergent,

$$u(x, y) = \sum_{k,\ell=1}^{\infty} \frac{c_{k\ell}}{k^2 + \ell^2} \sin kx \sin \ell y$$

is a solution of $-\Delta u = f$ with homogeneous Dirichlet boundary conditions. The preceding equivalences yield $u \in H^2([0, \pi]^2)$. \square

2.11 Let Ω be bounded with $\Gamma := \partial\Omega$, and let $g : \Gamma \rightarrow \mathbb{R}$ be a given function. Find the function $u \in H^1(\Omega)$ with minimal H^1 -norm which coincides with g on Γ . Under what conditions on g can this problem be handled in the framework of this section?

Solution. Let g be the restriction of a function $u_1 \in C^1(\bar{\Omega})$. We look for $u \in H_0^1(\Omega)$ such that $\|u_1 + u\|_1$ is minimal. This variational problem is solved by

$$(\nabla u, \nabla v)_0 + (u, v)_0 = \langle \ell, v \rangle \quad \forall v \in H_0^1$$

with $\langle \ell, v \rangle := -(\nabla u_1, \nabla v)_0 - (u_1, v)_0$.

It is the topic of the next § to relax the conditions on the boundary values. \square

2.12 Consider the elliptic, but not uniformly elliptic, bilinear form

$$a(u, v) := \int_0^1 x^2 u' v' dx$$

on the interval $[0, 1]$. Show that the problem $J(u) := \frac{1}{2}a(u, u) - \int_0^1 u dx \rightarrow \min!$ does not have a solution in $H_0^1(0, 1)$. – What is the associated (ordinary) differential equation?

Solution. We start with the solution of the associated differential equation

$$-\frac{d}{dx} x^2 \frac{d}{dx} u = 1.$$

First we require only the boundary condition at the right end, i.e., $u(1) = 0$, and obtain with the free parameter A :

$$u(x) = -\log x + A\left(\frac{1}{x} - 1\right).$$

When we restrict ourselves to the subinterval $[\delta, 1]$ with $\delta > 0$ and require $u_\delta(\delta) = 0$, the (approximate) solution is

$$u_\delta(x) = -\log x + \frac{\delta \log \delta}{1 - \delta} \left(\frac{1}{x} - 1\right)$$

for $x > \delta$ and $u_\delta(x) = 0$ for $0 \leq x \leq \delta$. Note that $\lim_{\delta \rightarrow 0} u_\delta(x) = -\log x$ for each $x > 0$.

Elementary calculations show that $\lim_{\delta \rightarrow 0} J(u_\delta) = J(-\log x)$ and that $\|u_\delta\|_1$ is unbounded for $\delta \rightarrow 0$. There is no solution in $H_0^1(0, 1)$ although the functional J is bounded from below.

We emphasize another consequence. Due to Remark II.1.8 $H^1[a, b]$ is embedded into $C[a, b]$, but $\int_0^1 x^2 v'(x)^2 dx < \infty$ does not imply the continuity of v . \square

2.14 In connection with Example 2.7, consider the continuous linear mapping

$$\begin{aligned} L : \ell_2 &\rightarrow \ell_2, \\ (Lx)_k &= 2^{-k}x_k. \end{aligned}$$

Show that the range of L is not closed.

Hint: The closure contains the point $y \in \ell_2$ with $y_k = 2^{-k/2}$, $k = 1, 2, \dots$

Solution. Following the hint define the sequence $\{x^j\}$ in ℓ_2 by

$$x_k^j = \begin{cases} 2^{+k/2} & \text{if } j \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

From $y = \lim_{j \rightarrow \infty} Lx^j$ it follows that y belongs to the closure of the range, but there is no $x \in \ell_2$ with $Lx = y$. \square

3.7 Suppose the domain Ω has a piecewise smooth boundary, and let $u \in H^1(\Omega) \cap C(\bar{\Omega})$. Show that $u \in H_0^1(\Omega)$ is equivalent to $u = 0$ on $\partial\Omega$.

Solution. Instead of performing a calculation as in the proof of the trace theorem, we will apply the trace theorem directly.

Let $u \in H_0^1(\Omega) \cap C(\bar{\Omega})$ and suppose that $u(x_0) \neq 0$ for some $x_0 \in \Gamma$. There is a smooth part $\Gamma_1 \subset \Gamma$ with $x_0 \in \Gamma_1$ and $|u(x)| \geq \frac{1}{2}|u(x_0)|$ for $x \in \Gamma_1$. In particular, $\|u\|_{0,\Gamma_1} \neq 0$. By definition of $H_0^1(\Omega)$ there is a sequence $\{v_n\}$ in $C_0^\infty(\Omega)$ that converges to u . Clearly, $\|v_n\|_{0,\Gamma_1} = 0$ holds for all n , and $\lim_{n \rightarrow \infty} \|v_n\|_{0,\Gamma_1} = 0 \neq \|u\|_{0,\Gamma_1}$. This contradicts the continuity of the trace operator. We conclude from the contradiction that $u(x_0) = 0$. \square

4.4 As usual, let u and u_h be the functions which minimize J over V and S_h , respectively. Show that u_h is also a solution of the minimum problem

$$a(u - v, u - v) \longrightarrow \min_{v \in S_h} !$$

Because of this, the mapping

$$\begin{aligned} R_h : V &\longrightarrow S_h \\ u &\longmapsto u_h \end{aligned}$$

is called the *Ritz projector*.

Solution. Given $v_h \in S_h$, set $w_h := v_h - u_h$. From the Galerkin orthogonality (4.7) and the symmetry of the bilinear form we conclude with the Binomial formula that

$$\begin{aligned} a(u - v_h, u - v_h) &= a(u - u_h, u - u_h) + 2a(u - u_h, w_h) + a(w_h, w_h) \\ &= a(u - u_h, u - u_h) + a(w_h, w_h) \\ &\geq a(u - u_h, u - u_h). \end{aligned}$$

This proves that the minimum is attained at u_h . \square

4.6 Suppose in Example 4.3 that on the bottom side of the square we replace the Dirichlet boundary condition by the natural boundary condition $\partial u/\partial \nu = 0$. Verify that this leads to the stencil

$$\begin{bmatrix} & -1 & \\ -1/2 & 2 & -1/2 \\ & & \end{bmatrix}_*$$

at these boundary points.

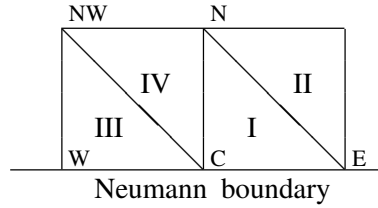


Fig. Numbering of the elements next to the center C on the Neumann boundary.

Solution. Let C be a point on the Neumann boundary. The boundary condition $\partial u/\partial \nu = 0$ is a natural boundary condition for the Poisson equation, and it is incorporated by testing u with the finite element functions in H^1 and not only in H_0^1 . Specifically, it is tested with the nodal function ψ_C that lives on the triangles I–IV in the figure above. Recalling the computations in Example 4.3 we get

$$\begin{aligned} a(\psi_C, \psi_C) &= \int_{I-IV} (\nabla \psi_C)^2 dx dy \\ &= \int_{I+III+IV} [(\partial_1 \psi_C)^2 + (\partial_2 \psi_C)^2] dx dy \\ &= \int_{I+III} (\partial_1 \psi_C)^2 dx dy + \int_{I+IV} (\partial_2 \psi_C)^2 dx dy \\ &= h^{-2} \int_{I+III} dx dy + h^{-2} \int_{I+IV} dx dy = 2, \end{aligned}$$

There is no change in the evaluation of the bilinear form for the nodal function associated to the point north of C , i.e., $a(\psi_C, \psi_N) = -1$. Next we have

$$\begin{aligned} a(\psi_C, \psi_E) &= \int_I \nabla \psi_C \cdot \nabla \psi_E dx dy \\ &= \int_I \partial_1 \psi_C \partial_1 \psi_E dx dy = \int_I (-h^{-1}) h^{-1} dx dy = -1/2. \end{aligned}$$

Since the same number is obtained for $a(\psi_C, \psi_W)$, the stencil is as given in the problem. \square

5.14 The completion of the space of vector-valued functions $C^\infty(\Omega)^n$ w.r.t. the norm

$$\|v\|^2 := \|v\|_{0,\Omega}^2 + \|\operatorname{div} v\|_{0,\Omega}^2$$

is denoted by $H(\operatorname{div}, \Omega)$. Obviously, $H^1(\Omega)^n \subset H(\operatorname{div}, \Omega) \subset L_2(\Omega)^n$. Show that a piecewise polynomial v is contained in $H(\operatorname{div}, \Omega)$ if and only if the components $v \cdot \nu$ in the direction of the normals are continuous on the inter-element boundaries. Hint: Apply Theorem 5.2 and use (2.22). — Similarly piecewise polynomials in the space $H(\operatorname{rot}, \Omega)$ are characterized by the continuity of the tangential components; see Problem VI.4.8.

Solution. By definition, $w = \operatorname{div} v$ holds in the weak sense if

$$\int_{\Omega} w \phi dx = - \int_{\Omega} v \cdot \nabla \phi dx \quad \forall \phi \in C_0^\infty(\Omega). \quad (1)$$

Assume that $\Omega = \Omega_1 \cup \Omega_2$ and that $v|_{\Omega_i} \in C^1(\Omega_i)$ for $i = 1, 2$. Set $\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$. By applying Green's formula to the subdomains we obtain

$$\begin{aligned} - \int_{\Omega} v \cdot \nabla \phi dx &= - \sum_{i=1}^2 \int_{\Omega_i} v \cdot \nabla \phi dx \\ &= \sum_{i=1}^2 \left[\int_{\Omega_i} \operatorname{div} v \phi dx + \int_{\partial\Omega_i} v \cdot \nu \phi dx \right] \\ &= \int_{\Omega} \operatorname{div} v \phi dx + \int_{\Gamma_{12}} [v \cdot \nu] \phi dx. \end{aligned} \quad (2)$$

Here $[\cdot]$ denotes the jump of a function. The right-hand side of (2) can coincide with the left-hand side of (1) for all $\phi \in C_0^\infty$ only if the jump of the normal component vanishes.

Conversely, if the jumps of the normal component vanish, then (1) holds if we set pointwise $w(x) := \operatorname{div} v(x)$, and this function is the divergence in the weak sense. \square

6.12 Let \mathcal{T}_h be a family of uniform partitions of Ω , and suppose S_h belong to an affine family of finite elements. Suppose the nodes of the basis are z_1, z_2, \dots, z_N with $N = N_h = \dim S_h$. Verify that for some constant c independent of h , the following inequality holds:

$$c^{-1} \|v\|_{0,\Omega}^2 \leq h^2 \sum_{i=1}^N |v(z_i)|^2 \leq c \|v\|_{0,\Omega}^2 \quad \text{for all } v \in S_h.$$

Solution. Let $\hat{z}_1, \hat{z}_2, \dots, \hat{z}_s$ be the nodes of a basis of the s -dimensional space Π on the reference triangle T_{ref} . The norm

$$\|v\| := \left(\sum_{i=1}^s |v(\hat{z}_i)|^2 \right)^{1/2}$$

is equivalent to $\|\cdot\|_{0, T_{\text{ref}}}$ on Π since Π is a finite dimensional space. Let T_h be an element of \mathcal{T}_h with diameter h . A scaling argument in the spirit of the transformation formula 6.6 shows that

$$\|v\|_{0, T_h} \quad \text{and} \quad h^2 \sum_{z_i \in T_h} |v(z_i)|^2$$

differ only by a factor that is independent of h . By summing over all elements of the triangulation we obtain the required formula. \square

6.13 Under appropriate assumptions on the boundary of Ω , we showed that

$$\inf_{v \in S_h} \|u - v_h\|_{1, \Omega} \leq c h \|u\|_{2, \Omega},$$

where for every $h > 0$, S_h is a finite-dimensional finite element space. Show that this implies the compactness of the imbedding $H^2(\Omega) \hookrightarrow H^1(\Omega)$. [Thus, the use of the compactness in the proof of the approximation theorem was not just a coincidence.]

Solution. Let B be the unit ball in $H^2(\Omega)$.

Let $\varepsilon > 0$. Choose h such that $ch < \varepsilon/4$, and for any $u \in B$ we find $v_h \in S_h$ with $\|u - v_h\|_1 \leq \varepsilon/4$. Since $\dim S_h$ is finite, the bounded set $\{v \in S_h; \|v\|_1 \leq 1\}$ can be covered by a finite number of balls with diameter $\varepsilon/2$. If the diameter of these balls are doubled, they cover the set B . Hence, B is precompact, and the completeness of the Sobolev space implies compactness. \square

7.11 Let $\Omega = (0, 2\pi)^2$ be a square, and suppose $u \in H_0^1(\Omega)$ is a weak solution of $-\Delta u = f$ with $f \in L_2(\Omega)$. Using Problem 1.16, show that $\Delta u \in L_2(\Omega)$, and then use the Cauchy–Schwarz inequality to show that all second derivatives lie in L_2 , and thus u is an H^2 function.

Solution. We rather let $\Omega = (0, \pi)^2$ since this does not change the character of the problem.

We extend f to $\Omega_{\text{sym}} := (-\pi, \pi)^2$ by the (anti-) symmetry requirements

$$f(-x, y) = -f(x, y), \quad f(x, -y) = -f(x, y),$$

without changing the symbol. Since $f \in L_2(\Omega_{sym})$, f can be represented as a Fourier series with sine functions only

$$f(x, y) = \sum_{k, \ell=1}^{\infty} a_{k\ell} \sin kx \sin \ell y.$$

Parseval's inequality yields

$$\sum_{k, \ell} |a_{k\ell}|^2 = \pi^2 \|f\|_{2, \Omega}^2.$$

Obviously, the solution has the representation

$$u(x, y) = \sum_{k, \ell} \frac{a_{k\ell}}{k^2 + \ell^2} \sin kx \sin \ell y.$$

The coefficients in the representation

$$u_{xx} = - \sum_{k, \ell} \frac{k^2}{k^2 + \ell^2} a_{k\ell} \sin kx \sin \ell y$$

are obviously square summable, and $u_{xx} \in L_2(\Omega)$. The same is true for u_{yy} . More interesting is

$$u_{xy} = \sum_{k, \ell} \frac{k\ell}{k^2 + \ell^2} a_{k\ell} \cos kx \cos \ell y.$$

From Young's inequality $2k\ell \leq k^2 + \ell^2$ we conclude that we have square summability also here. Hence, $u_{xy} \in L_2(\Omega)$, and the proof of $u \in H^2(\Omega)$ is complete. \square

Chapter III

1.12 The Crouzeix–Raviart element has locally the same degrees of freedom as the conforming P_1 element \mathcal{M}_0^1 , i. e. the Courant triangle. Show that the (global) dimension of the finite element spaces differ by a factor that is close to 3 if a rectangular domain as in Fig. 9 is partitioned.

Solution. The nodal variables of the conforming P_1 element are associated to the nodes of a mesh (as in Fig. 9) with mesh size h .

The nodal points of the corresponding nonconforming P_1 element are associated to the mesh with meshsize $h/2$, but with those of the h -mesh excluded. Since halving the meshsize induces a factor of about 4 in the number of points, the elimination of the original points gives rise to a factor of about 3. \square

3.8 Let $a : V \times V \rightarrow \mathbb{R}$ be a positive symmetric bilinear form satisfying the hypotheses of Theorem 3.6. Show that a is elliptic, i.e., $a(v, v) \geq \alpha_1 \|v\|_V^2$ for some $\alpha_1 > 0$.

Solution. Given u , by the inf-sup condition there is a $v \neq 0$ such that $\frac{1}{2}\alpha \|u\|_V \leq a(u, v)/\|v\|_V$. The Cauchy inequality and (3.6) yield

$$\frac{1}{4}\alpha^2 \|u_h\|_V^2 \leq \frac{a(u, v)^2}{\|v\|_V^2} \leq a(u, u) \frac{a(v, v)}{\|v\|_V^2} \leq Ca(u, u).$$

Therefore, we have ellipticity with $\alpha_1 \geq \alpha^2/(4C)$.

3.9 [Nitsche, private communication] Show the following converse of Lemma 3.7: Suppose that for every $f \in V'$, the solution of (3.5) satisfies

$$\lim_{h \rightarrow 0} u_h = u := L^{-1}f.$$

Then

$$\inf_h \inf_{u_h \in U_h} \sup_{v_h \in V_h} \frac{a(u_h, v_h)}{\|u_h\|_U \|v_h\|_V} > 0.$$

Hint: Use (3.10) and apply the principle of uniform boundedness.

Solution. Given $f \in V'$, denote the solution of (3.5) by u_h . Let $K_h : V' \rightarrow U_h \subset U$ be the mapping that sends f to u_h . Obviously, K_h is linear. To be precise, we assume that u_h is always well defined. Since $\|f|_{V'_h}\|_{V'} \leq \|f\|_{V'}$, each K_h is a bounded linear mapping. From $\lim_{h \rightarrow 0} K_h f = L^{-1}f$ we conclude that $\sup_h \|K_h f\| < \infty$ for each $f \in V'$. The principle of uniform boundedness assures that

$$\alpha^{-1} := \sup_h \|K_h\| < \infty.$$

Hence, $\|K_h u_h\| \geq \alpha \|u_h\|$ holds for each $u_h \in V'$. Finally, the equivalence of (3.7) and (3.10) yields the inf-sup condition with the uniform bound $\alpha > 0$. \square

3.10 Show that

$$\begin{aligned} \|v\|_0^2 &\leq \|v\|_m \|v\|_{-m} && \text{for all } v \in H_0^m(\Omega), \\ \|v\|_1^2 &\leq \|v\|_0 \|v\|_2 && \text{for all } v \in H^2(\Omega) \cap H_0^1(\Omega). \end{aligned}$$

Hint: To prove the second relation, use the Helmholtz equation $-\Delta u + u = f$.

Solution. By definition II.3.1 we have

$$(u, v)_0 \leq \|u\|_{-m} \|v\|_m.$$

Setting $u := v$ we obtain $\|v\|_0^2 \leq \|v\|_{-m} \|v\|_m$, i.e., the first statement.

Since zero boundary conditions are assumed, Green's formula yields

$$\int_{\Omega} w_i \partial_i v ds = - \int_{\Omega} \partial_i w_i v dx.$$

Setting $w_i := \partial_i v$ and summing over i we obtain

$$\int_{\Omega} \nabla v \cdot \nabla v dx = - \int_{\Omega} \Delta v v dx.$$

With the Cauchy inequality and $\|\nabla v\|_0 \leq \|v\|_2$ the inequality for $s = 1$ is complete. \square

4.16 Show that the inf-sup condition (4.8) is equivalent to the following decomposition property: For every $u \in X$ there exists a decomposition

$$u = v + w$$

with $v \in V$ and $w \in V^\perp$ such that

$$\|w\|_X \leq \beta^{-1} \|Bu\|_{M'},$$

where $\beta > 0$ is a constant independent of u .

Solution. This problem is strongly related to Lemma 4.2(ii). Assume that (4.8) holds. Given $u \in X$, since V and V^\perp are closed, there exists an orthogonal decomposition

$$u = v + w, \quad v \in V, w \in V^\perp. \quad (1)$$

From Lemma 4.2(ii) it follows that $\|Bw\|_{M'} \geq \beta \|w\|_X$. Since v in the decomposition (1) lies in the kernel of B , we have $\|w\|_X \leq \beta^{-1} \|Bw\|_{M'} = \beta^{-1} \|Bu\|_{M'}$.

Conversely, assume that the decomposition satisfies the conditions as formulated in the problem. If $u \in V^\perp$, then $v = 0$ and $\|u\|_X \leq \beta^{-1} \|Bu\|_{M'}$ or $\|Bu\|_{M'} \geq \beta \|u\|_X$. Hence, the statement in Lemma 4.2(ii) is verified. \square

4.21 The pure Neumann Problem (II.3.8)

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= g \quad \text{on } \partial\Omega \end{aligned}$$

is only solvable if $\int_{\Omega} f dx + \int_{\Gamma} g ds = 0$. This compatibility condition follows by applying Gauss' integral theorem to the vector field ∇u . Since $u + \text{const}$ is a solution whenever u is, we can enforce the constraint

$$\int_{\Omega} u dx = 0.$$

Formulate the associated saddle point problem, and use the trace theorem and the second Poincaré inequality to show that the hypotheses of Theorem 4.3 are satisfied.

Solution. Consider the saddle-point problem with $X = H^1(\Omega)$, $M = \mathbb{R}$, and the bilinear forms

$$\begin{aligned} a(u, v) &= \int_{\Omega} \nabla u \nabla v dx, \\ b(u, \lambda) &= \lambda \int_{\Omega} v dx = \lambda \bar{v} \mu(\Omega). \end{aligned}$$

Adopt the notation of Problem II.1.12. With the variant of Friedrich's inequality there we obtain

$$\begin{aligned} \|v\|_1^2 &= |v|_1^2 + \|v\|_0^2 \\ &\leq |v|_1^2 + 2c^2(|\bar{v}|^2 + |v|_1^2) \\ &\leq c^1[a(v, v) + |\bar{v}|^2] \\ &= c^1 a(v, v) \quad \text{if } \bar{v} = 0. \end{aligned}$$

This proves ellipticity of $a(\cdot, \cdot)$ on the kernel.

The inf-sup condition is verified by taking the constant test function $v_0 = 1$:

$$b(\lambda, v_0) = \lambda \int_{\Omega} dx = \lambda \mu(\Omega) = \lambda \|v_0\|_0 \mu(\Omega)^{1/2} = \lambda \|v_0\|_1 \mu(\Omega)^{1/2}.$$

The condition holds with the constant $\mu(\Omega)^{1/2}$. \square

4.22 Let a, b , and c be positive numbers. Show that $a \leq b + c$ implies that $a \leq b^2/a + 2c$.

Solution.

$$a \leq b(b+c)/(b+c) + c = b^2/(b+c) + c(1+b/(b+c)) \leq b^2/a + 2c. \quad \square$$

6.8 [7.4 in 2nd ed.] If Ω is convex or sufficiently smooth, then one has for the Stokes problem the regularity result

$$\|u\|_2 + \|p\|_1 \leq c \|f\|_0; \quad (7.18)$$

see Girault and Raviart [1986]. Show by a duality argument the L_2 error estimate

$$\|u - u_h\|_0 \leq ch(\|u - u_h\|_1 + \|p - p_h\|_0). \quad (7.19)$$

Solution. As usually in duality arguments consider an auxiliary problem. Find $\varphi \in X, r \in M$ such that

$$\begin{aligned} a(w, \varphi) + b(w, r) &= (u - u_0, w)_0 \quad \text{for all } w \in X, \\ b(\varphi, q) &= 0 \quad \text{for all } q \in M. \end{aligned} \quad (1)$$

The regularity assumption yields $\|\varphi\|_2 + \|r\|_1 \leq C\|u - u_0\|_0$, and by the usual approximation argument there are $\varphi_h \in X_h, r_h \in M_h$ such that

$$\|\varphi - \varphi_h\|_1 + \|r - r_h\|_0 \leq Ch\|u - u_0\|_0.$$

The subtraction of (4.4) and (4.5) with the test function φ_h, r_h yields the analogon to Galerkin orthogonality

$$\begin{aligned} a(u - u_h, \varphi_h) + b(\varphi_h, p - p_h) &= 0, \\ b(u - u_h, r_h) &= 0. \end{aligned}$$

Now we set $w := u - u_h, q := p - p_h$ in (1) and obtain

$$\begin{aligned} (u - u_h, u - u_h)_0 &= a(u - u_h, \varphi) + b(u - u_h, r) + b(\varphi, p - p_h) \\ &= a(u - u_h, \varphi - \varphi_h) + b(u - u_h, r - r_h) + b(\varphi - \varphi_h, p - p_h) \\ &\leq C(\|u - u_h\|_1 \|\varphi - \varphi_h\|_1 + \|u - u_h\|_1 \|r - r_h\|_1 + \|\varphi - \varphi_h\|_1 \|p - p_h\|_0) \\ &\leq C(\|u - u_h\|_1 + \|u - u_h\|_1 + \|p - p_h\|_0) h \|u - u_h\|_0. \end{aligned}$$

After dividing by $\|u - u_h\|_0$ the proof is complete. \square

Chapter IV

2.6 By (2.5), $\alpha_k \geq \alpha^* := 1/\lambda_{\max}(A)$. Show that convergence is guaranteed for every fixed step size α with $0 < \alpha < 2\alpha^*$.

Solution. We perform a spectral decomposition of the error

$$x_k - x^* = \sum_{j=1}^n \beta_j z_j$$

with $Az_j = \lambda_j z_j$ for $j = 1, \dots, n$. The iteration

$$x_{k+1} = x_k + \alpha(b - Ax_k)$$

leads to

$$x_{k+1} - x^* = (1 - \alpha A)(x_k - x^*) = \sum_{j=1}^n (1 - \alpha \lambda_j) \beta_j z_j.$$

The damping factors satisfy $-1 < 1 - \alpha \lambda_j < 1$ if $0 < \alpha < 2/\lambda_{\max}(A)$, and convergence is guaranteed. \square

4.8 Show that the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

is positive definite, and that its condition number is 4.

Hint: The quadratic form associated with the matrix A is $x^2 + y^2 + z^2 + (x + y + z)^2$.

Solution. The formula in the hint shows that $A \geq I$. By applying Young's inequality to the nondiagonal terms, we see that $(x + y + z)^2 \leq 3(x^2 + y^2 + z^2)$ and $A \leq 4I$. The quotient of the factors in the upper and the lower bound is 4. \square

4.14 Let $A \leq B$ denote that $B - A$ is positive semidefinite. Show that $A \leq B$ implies $B^{-1} \leq A^{-1}$, but it does not imply $A^2 \leq B^2$. — To prove the first part note that $(x, B^{-1}x) = (A^{-1/2}x, A^{1/2}B^{-1}x)$ and apply Cauchy's inequality. Next consider the matrices

$$A := \begin{pmatrix} 1 & a \\ a & 2a^2 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 2 & 0 \\ 0 & 3a^2 \end{pmatrix}$$

for establishing the negative result. From the latter it follows that we cannot derive good preconditioners for the biharmonic equation by applying those for the poisson equation twice.

Note: The converse is more favorable, i.e., $A^2 \leq B^2$ implies $A \leq B$. Indeed, the Rayleigh quotient $\lambda = \max\{(x, Ax)/(x, Bx)\}$ is an eigenvalue, and the maximum is attained at an eigenvector, i.e., $Ax = \lambda Bx$. On the other hand, by assumption

$$0 \leq (x, B^2x) - (x, A^2x) = (1 - \lambda^2) \|Bx\|^2.$$

Hence, $\lambda \leq 1$ and the proof is complete.

Solution. By Cauchy's inequality and $A \leq B$ it follows that

$$\begin{aligned} (x, B^{-1}x)^2 &= (A^{-1/2}x, A^{1/2}B^{-1}x)^2 \leq (x, A^{-1}x) (B^{-1}x, AB^{-1}x) \\ &\leq (x, A^{-1}x) (B^{-1}x, BB^{-1}x). \end{aligned}$$

We divide by $(x, B^{-1}x)$ and obtain $B^{-1} \leq A^{-1}$.

Consider the given matrices. The relation $(x, Ax) \leq (x, Bx)$ is established by applying Young's inequality to the nondiagonal terms. Furthermore

$$A^2 = \begin{pmatrix} 1 + a^2 & a + 2a^3 \\ a + 2a^3 & a^2 + 4a^4 \end{pmatrix}, \quad B^2 = \begin{pmatrix} 4 & 0 \\ 0 & 9a^4 \end{pmatrix}.$$

Obviously $B^2 - A^2$ has a negative diagonal entry if $a \geq 2$. \square

4.15 Show that $A \leq B$ implies $B^{-1}AB^{-1} \leq B^{-1}$.

Solution. If $x = B^{-1}z$, then $(x, Ax) \leq (x, Bx)$ reads
 $(B^{-1}z, AB^{-1}z) \leq (B^{-1}z, BB^{-1}z)$, i.e., $(z, B^{-1}AB^{-1}z) \leq (z, B^{-1}z)$. \square

4.16 Let A and B be symmetric positive definite matrices with $A \leq B$. Show that

$$(I - B^{-1}A)^m B^{-1}$$

is positive definite for $m = 1, 2, \dots$. To this end note that

$$q(XY)X = Xq(YX)$$

holds for any matrices X and Y if q is a polynomial. Which assumption may be relaxed if m is even?

Remark: We can only show that the matrix is semidefinite since $A = B$ is submitted by the assumptions.

Solution. First let m be an even number, $m = 2n$. We compute

$$\begin{aligned} (x, (I - B^{-1}A)^{2n} B^{-1}x) &= (x, (I - B^{-1}A)^n B^{-1}(I - AB^{-1})^n x) \\ &= ((I - AB^{-1})^n x, B^{-1}(I - AB^{-1})x) \\ &= (z, B^{-1}z) \geq 0, \end{aligned}$$

where $z := (I - AB^{-1})^n x$. This proves that the matrix is positive semidefinite. [Here we have only used that B is invertible.]

Similar we get with z as above

$$\begin{aligned} (x, (I - B^{-1}A)^{2n+1} B^{-1}x) &= (z, B^{-1}(I - AB^{-1})z) \\ &= (z, (B^{-1} - B^{-1}AB^{-1})z). \end{aligned}$$

The preceding problem made clear that $B^{-1} - B^{-1}AB^{-1} \geq 0$. \square

Chapter V

2.11 Show that for the scale of the Sobolev spaces, the analog

$$\|v\|_{s,\Omega}^2 \leq \|v\|_{s-1,\Omega} \|v\|_{s+1,\Omega}$$

of (2.5) holds for $s = 0$ and $s = 1$.

For the solution look at Problem III.3.10. \square

Chapter VI

6.11 Show that

$$\|\operatorname{div} \eta\|_{-1} \leq \operatorname{const} \sup_{\gamma} \frac{(\gamma, \eta)_0}{\|\gamma\|_{H(\operatorname{rot}, \Omega)}},$$

and thus that $\operatorname{div} \eta \in H^{-1}(\Omega)$ for $\eta \in (H_0(\operatorname{rot}, \Omega))'$. Since $H_0(\operatorname{rot}, \Omega) \supset H_0^1(\Omega)$ implies $(H_0(\operatorname{rot}, \Omega))' \subset H^{-1}(\Omega)$, this completes the proof of (6.9).

Solution. Let $v \in H_0^1(\Omega)$. Its gradient $\gamma := \nabla v$ satisfies $\nabla v \cdot \tau = 0$ on $\partial\Omega$. Since $\operatorname{rot} \nabla v = 0$, we have $\gamma \in H_0(\operatorname{rot}, \Omega)$ and $\|\gamma\|_0 = \|\gamma\|_{H_0(\operatorname{rot}, \Omega)}$. Partial integration yields

$$\begin{aligned} \|\operatorname{div} \eta\|_{-1} &= \sup_{v \in H_0^1(\Omega)} \frac{(v, \operatorname{div} \eta)_0}{\|v\|_1} \\ &= \sup_{v \in H_0^1(\Omega)} \frac{(\nabla v, \eta)_0}{(\|\nabla v\|_0^2 + \|v\|_0^2)^{1/2}} \\ &\leq \sup_{\gamma} \frac{(\gamma, \eta)_0}{\|\gamma\|_{H_0(\operatorname{rot}, \Omega)}}. \end{aligned}$$

A standard density argument yields $\operatorname{div} \eta \in H^{-1}(\Omega)$.