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# Boundedness properties of fermionic operators

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The fermionic second quantization operator $d\Gamma(B)$ is shown to be bounded by a power $N^{s/2}$ of the number operator $N$ given that the operator $B$ belongs to the $r$th von Neumann–Schatten class, $s=2(r-1)/r$. Conversely, number operator estimates for $d\Gamma(B)$ imply von Neumann–Schatten conditions on $B$. Quadratic creation and annihilation operators are treated as well. © *2010 American Institute of Physics.* [doi:10.1063/1.3464264]

## I. INTRODUCTION

Operators that satisfy the canonical anticommutation relations (CAR) are necessarily bounded. One may therefore ask what can be said about more complicated operators, say, quadratic expressions in creation and annihilation operators. Perhaps the most prominent such operator is $d\Gamma(B)$, the functor of second quantization.

Suppose, we are given a Fock representation of the CAR over a separable complex Hilbert space $L$. With the usual annihilation and creation operators $a(f)$ and $a^\dagger(f)$, we define for a bounded operator $B$ on $L$ its second quantization through

$$d\Gamma(B) \coloneqq \sum_j a^\dagger(Be_j)a(\bar{e}_j), \tag{1}$$

where $\{e_j\}$ is a complete orthonormal system (ONS) in $L$. The details of this construction are briefly described in Sec. II In general, $d\Gamma(B)$ is an unbounded operator and its degree of unboundedness is best measured by powers $N^\alpha$ of the number operator,

$$N \coloneqq d\Gamma(\mathbb{1}) = \sum_j a^\dagger(e_j)a(\bar{e}_j).$$

Intuitively, one would expect that a quadratic operator can only be bounded by another quadratic operator $N$, which is correct for bosons. Thanks to the fermionic character, however, there is an almost equivalent relation between the exponent $\alpha$ and the regularity of $B$ by which we mean $B \in B_r(L)$, where $B_r(L)$ is a von Neumann–Schatten class. Theorem III.5 tells us

$$d\Gamma(B)^* d\Gamma(B) \le \begin{cases} \|B\|_r^2 N^s + \|B\|_2^2 \mathbb{1} & 1<r<2 \\ \|B\|_r^2 N^s & r=1, \quad 2 \le r \le \infty, \end{cases} \tag{2}$$

whenever $B \in B_r(L)$, $1 \le r \le \infty$, and $s=\frac{2(r-1)}{r}$. Hence, when $B$ is more regular so is $d\Gamma(B)$. Estimate (2) is proven by a thorough analysis of (1) involving Hölder and Cauchy–Schwarz inequalities for operators.

In the literature only the cases $s=0$ ($r=1$) and $s=2$ ($r=\infty$) are known. By using the $s=2$ estimate, Carey and Ruijsenaars[3] as well as Grosse and Langmann[5] showed that the exponential series $\exp(zd\Gamma(B))$ converges strongly when $z$ is from some bounded disk. For $s<2$ strong convergence immediately holds for all $z \in \mathbb{C}$ (see Lemma IV.2 for related calculations). Furthermore, estimate (2) guarantees that series (1) converges strongly on the domain $D(N^{s/2})$.

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In Sec. IV, Theorem IV.1 answers the question as to how boundedness properties of $d\Gamma(B)$ affect the corresponding operator $B$ which is only interesting for $\dim L=\infty$. Its proof uses only elementary calculations. For $s>0$ it turns out that in a way bound (2) is almost sharp. That is to say, an estimate with $N^s$ implies $B\in B_{r+\varepsilon}(L)$ for all $\varepsilon>0$. For $s=0$ we may even forget about $\varepsilon$ in that an estimate with $s=0$ implies $B\in B_1(L)$ which was conjectured by Carey and Ruijsenaars[3] and Ottesen.[7] It is an open question whether one may drop $\varepsilon$ altogether.

All theorems proved for $d\Gamma(B)$ have analogs for the quadratic annihilation and creation operators,

$$\Delta(A) := \sum_j a(Ae_j)a(\bar{e}_j), \quad \Delta^+(C) := \sum_j a^\dagger(Ce_j)a^\dagger(\bar{e}_j). \tag{3}$$

Theorems III.6 and III.7 present number operator estimates in the spirit of (2) for $1\le r\le 2$ since $\Delta(A)$ and $\Delta^+(C)$ are well-defined only for $A,C\in B_2(L)$. Hence, the $s=2$ ($r=\infty$) estimates from the literature, see (21), are far from optimal. The proofs parallel that for $d\Gamma(B)$. Contrary to that, the converse Theorems IV.5 and IV.4 are not elementary but employ a determinant formula for fermionic Gaussians and a theorem from complex analysis. Their statement is essentially the same as for $d\Gamma(B)$ except for the case $r=1$ which also has an $\varepsilon>0$.

## II. THE CAR AND SECOND QUANTIZATION

We sketch the necessary background from fermionic Fock space theory. Presentations similar in spirit can be found in the study of Carey and Ruijsenaars[3] and Ottesen.[7] We formulate the CAR for operator-valued functionals. To this end, let $L$ be a complex Hilbert space equipped with a conjugation $f\mapsto\bar{f}$. Throughout, we will assume $L$ to be separable. Let further $\mathcal{F}$ be another complex Hilbert space. We call a linear map from $L$ into the linear operators on $\mathcal{F}$,

$$f\in L, \quad f\mapsto c(f),$$

an operator-valued functional. The CAR need two such functionals, $a$ and $a^\dagger$, which are assumed to have a common dense domain of definition $D\subset\mathcal{F}$ and

$$a(f)D\subset D, \quad a^\dagger(f)D\subset D.$$

These operators are said to give a representation of the CAR if for all $f,g\in L$ on $D$,

$$\{a(f),a(g)\}=0=\{a^\dagger(f),a^\dagger(g)\}, \tag{4}$$

$$\{a(f),a^\dagger(g)\}=(\bar{f},g)\mathbb{1}, \tag{5}$$

where the curly brackets denote the anticommutator. We further require the unitarity condition

$$a(f)^*=a^\dagger(\bar{f}). \tag{6}$$

Properties (4)–(6) imply

$$(a^\dagger(f)a(\bar{f}))^2=\|f\|^2a^\dagger(f)a(\bar{f}) \quad \text{and} \quad (a^\dagger(f)a(\bar{f}))^*=a^\dagger(f)a(\bar{f}). \tag{7}$$

In particular, $a^\dagger(f)a(\bar{f})$ is an orthogonal projection for $\|f\|=1$ and thus

$$0\le a(f)^*a(f)\le\|f\|^2\mathbb{1}, \quad 0\le a^\dagger(f)^*a^\dagger(f)\le\|f\|^2\mathbb{1}. \tag{8}$$

We have the fundamental boundedness result.

**Theorem II.1:** *The operators $a(f)$ and $a^\dagger(f)$ are bounded on their domain of definition and therefore extend to bounded operators on all of $\mathcal{F}$. We have*

$$\|a(f)\|=\|a^\dagger(f)\|=\|f\|. \tag{9}$$

*Hence, the maps* $f \mapsto a(f)$, $f \mapsto a^\dagger(f)$ *are continuous and injective.*

In what follows, we will work exclusively within the Fock representation. It features a special vector, the vacuum $\Omega \in \mathcal{F}$, $\|\Omega\| = 1$. It is annihilated by the $a(f)$'s,

$$a(f)\Omega = 0 \quad \text{for all} \;\; f \in L, \tag{10}$$

and cyclic for the $a^\dagger(f)$'s, i.e.,

$$\overline{\operatorname{span}\{a^\dagger(f_{j_n}) \cdots a^\dagger(f_{j_1})\Omega | n \in \mathbb{N}_0\}} = \mathcal{F}. \tag{11}$$

Consequently, $a(f)$ is called annihilation operator and $a^\dagger(f)$ creation operator. $\mathcal{F}$ is the Fock space. Because of the vacuum, the Fock space has a special structure which can be described best through the $n$-particle spaces,

$$\mathcal{F}^{(n)} \coloneqq \overline{\operatorname{span}\{a^\dagger(f_n) \cdots a^\dagger(f_1)\Omega\}}, \quad n \geq 0. \tag{12}$$

It is clear that $\mathcal{F}$ is built from these subspaces.

**Theorem II.2:** *The Fock space* $\mathcal{F}$ *is the (completed) orthogonal sum of the n-particle spaces* $\mathcal{F}^{(n)}$,

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)} \quad \text{with} \;\; \mathcal{F}^{(m)} \perp \mathcal{F}^{(n)}, \quad m \neq n.$$

In order to avoid running into technical difficulties, we will perform all calculations on the subspace of finite particle numbers,

$$\mathcal{F}_0 \coloneqq \operatorname{span}\{\Phi | \Phi \in \mathcal{F}^{(n)}, n \in \mathbb{N}_0\}. \tag{13}$$

Creation and annihilation operators are fully understood by Theorem II.1. The next more complicated operators are quadratic expressions in creators and annihilators. Such quadratic operators are used in second quantization as well as in constructing central extensions of certain Lie algebras. There are different methods of introducing them. Here we define them quite straightforwardly via the following series:

$$d\Gamma(B) \coloneqq \sum_j a^\dagger(Be_j)a(\bar{e}_j), \tag{14}$$

$$\Delta(A) \coloneqq \sum_j a(Ae_j)a(\bar{e}_j), \quad \Delta^+(C) \coloneqq \sum_j a^\dagger(Ce_j)a^\dagger(\bar{e}_j), \tag{15}$$

where $\{e_j\}$ is a complete ONS in $L$ and $A, B, C$ are linear operators on $L$. The operator $d\Gamma(B)$ gives the functor of second quantization. When $\dim L < \infty$ there is no problem of convergence. For general separable $L$ well-definedness can be shown under certain conditions at least on $\mathcal{F}_0$.

**Theorem II.3:** *Let* $B: L \to L$ *be bounded. Then,* $d\Gamma(B)$ *from (14) is well-defined on* $\mathcal{F}_0$ *and* $d\Gamma(B)^* = d\Gamma(B^*)$. *When* $B \geq 0$ *so is* $d\Gamma(B) \geq 0$. *Furthermore, let* $A, C: L \to L$ *be Hilbert–Schmidt operators with* $A^T = -A$ *and* $C^T = -C$, *where* $A^T \coloneqq \bar{A}^*$ *is the transpose. Then,* $\Delta(A)$ *and* $\Delta^+(C)$ *from (15) are well-defined on* $\mathcal{F}_0$ *and satisfy* $\Delta(A)^* = \Delta^+(A^*)$.

We will not touch upon the question as to whether the domain of definition can be enlarged. However, the conditions imposed on $A, B, C$ are in a way necessary. For $d\Gamma(B)$ to exist on the entire one-particle space $\mathcal{F}^{(1)}$, it is necessary that $B$ is bounded. Likewise, in order that $\Delta(A)$ exists on the entire two-particle subspace $\mathcal{F}^{(2)}$, it is necessary that $A$ is Hilbert–Schmidt. Finally, $\Delta^+(C)$ is defined on the vacuum only if $C$ is Hilbert–Schmidt.

We will need to know what $d\Gamma(B)$, $\Delta(A)$, and $\Delta^{+}(C)$ do with the $n$-particle spaces,

$$d\Gamma(B):\mathcal{F}^{(n)} \to \mathcal{F}^{(n)}, \quad \Delta(A):\mathcal{F}^{(n)} \to \mathcal{F}^{(n-2)}, \quad \Delta^{+}(C):\mathcal{F}^{(n)} \to \mathcal{F}^{(n+2)}. \tag{16}$$

That is why $\Delta(A)$ and $\Delta^{+}(C)$ are called quadratic annihilation and creation operators, respectively. $d\Gamma(B)$ preserves the number of particles. Of all the interesting algebraic properties, we only need one commutator,

$$[\Delta(A),\Delta^{+}(C)] = -4d\Gamma(CA) + 2 \operatorname{tr} AC \cdot \mathbb{1}. \tag{17}$$

By taking $B=\mathbb{1}$, we obtain the particle number operator or number operator for short

$$N := d\Gamma(\mathbb{1}) = \sum_j a^{\dagger}(e_j)a(\bar{e}_j).$$

We will use the commutators

$$[N,a(f)] = -a(f), \quad [N,a^{\dagger}(f)] = a^{\dagger}(f).$$

As an operator on the Fock space $N$ has a very simple structure,

$$N\Phi = n\Phi, \quad \Phi \in \mathcal{F}^{(n)}, \tag{18}$$

which justifies the naming. Moreover, $N$ is essentially self-adjoint on $\mathcal{F}_0$ and $N \geq 0$. Since $N$ as well as its functions are just multiples of the identity operator on each $\mathcal{F}^{(n)}$, they commute with number preserving operators.

## III. NUMBER OPERATOR ESTIMATES

We want to estimate $d\Gamma(B)$, $\Delta(A)$, and $\Delta^{+}(C)$ by the number operator $N$. The proofs usually rely on manipulating series, which are infinite when $\dim L=\infty$. This can always be justified by standard arguments based on partial sums. For the sake of the presentation's clarity, we will not carry this out. Furthermore, we write $B_r(L)$ for the $r$th von Neumann–Schatten class and $B_r^{-}(L)$ for the subset of skew-symmetric operators $A^T=-A$. Finally, for $1 \leq r < \infty$ we will employ the singular value decomposition,

$$A = \sum_j \mu_j(e_j,\cdot)f_j \tag{19}$$

with singular values $\mu_j \geq 0$ and ONS's $\{e_j\}$ and $\{f_j\}$. When not explicitly referring to (19), we mean $\{e_j\}$ to be any ONS.

To begin with, we cite a Jensen type inequality for operators. It goes back to Bhagwat and Subramanian.[2] See also Vasudeva and Singh[9] and Mond and Pečarić.[6]

*Proposition III.1: Let $w_j \in \mathbb{R}$, $w_j \geq 0$ for $j=1,\ldots,n$. Furthermore, let $c_j:\mathcal{H}\to\mathcal{H}$ be bounded non-negative operators on a Hilbert space $\mathcal{H}$. Then, for all $1 \leq p \leq q < \infty$,*

$$\left(\sum_{j=1}^{n} w_j c_j^p\right)^{1/p} \leq w^{1/p-1/q}\left(\sum_{j=1}^{n} w_j c_j^q\right)^{1/q}.$$

A simple consequence is a Hölder type inequality.

*Corollary III.2:* Let $\mu_j \in \mathbb{R}$, $\mu_j \geq 0$, for $j=1,\ldots,n$. Let furthermore $c_j:\mathcal{H}\to\mathcal{H}$ be bounded non-negative operators on a Hilbert space $\mathcal{H}$. Then, for $p,q\geq 1$, $\frac{1}{p}+\frac{1}{q}=1$,

$$\sum_{j=1}^{n} \mu_j c_j \leq \left(\sum_{j=1}^{n} \mu_j^p\right)^{1/p}\left(\sum_{j=1}^{n} c_j^q\right)^{1/q}.$$

*Proof:* First of all, we rewrite the Jensen inequality in Proposition III.1 for a special case,

$$\sum_{j=1}^{n} \mu_j c_j \leq \left(\sum_{j=1}^{n} \mu_j\right)^{1-1/q}\left(\sum_{j=1}^{n} \mu_j c_j^q\right)^{1/q}.$$

Without loss of generality we may assume $\mu_j>0$ for $j=1,\ldots,n$. Let $\frac{1}{p}+\frac{1}{q}=1$. Then,

$$\sum_{j=1}^{n} \mu_j c_j = \sum_{j=1}^{n} \frac{\mu_j^p}{\mu_j^{p-1}} c_j \leq \left(\sum_{j=1}^{n} \mu_j^p\right)^{1-1/q}\left(\sum_{j=1}^{n} \frac{\mu_j^p}{\mu_j^{(p-1)q}} c_j^q\right)^{1/q} = \left(\sum_{j=1}^{n} \mu_j^p\right)^{1/p}\left(\sum_{j=1}^{n} c_j^q\right)^{1/q},$$

which is Hölder's inequality. □

This allows us to treat a very special case.

*Lemma III.3: Let $\lambda_j \geq 0$. Assume*

$$\Lambda_p := \left(\sum_j \lambda_j^p\right)^{1/p} < \infty \quad \textit{for}\ 1\leq p<\infty \quad \textit{or} \quad \Lambda_\infty := \sup_j \lambda_j < \infty.$$

*Then, for $\frac{1}{p}+\frac{1}{q}=1$ and with the understanding $\frac{1}{\infty}=0$,*

$$\sum_j \lambda_j a^\dagger(e_j)a(\bar{e}_j) \leq \Lambda_p N^{1/q}.$$

*Proof:* The simplest cases are $p=1,\infty$. For $p=1$,

$$\sum_j \lambda_j a^\dagger(e_j)a(\bar{e}_j) \leq \sum_j \lambda_j \mathbb{1}$$

because of (8). For $p=\infty$,

$$\sum_j \lambda_j a^\dagger(e_j)a(\bar{e}_j) \leq \sup_j \lambda_j \sum_j a^\dagger(e_j)a(\bar{e}_j).$$

On to the cases $1<p<\infty$. By Hölder's inequality III.2,

$$\sum_j \lambda_j a^\dagger(e_j)a(\bar{e}_j) \leq \left(\sum_j \lambda_j^p\right)^{1/p}\left(\sum_j (a^\dagger(e_j)a(\bar{e}_j))^q\right)^{1/q} = \left(\sum_j \lambda_j^p\right)^{1/p} N^{1/q},$$

since, by (7), $a^\dagger(e_j)a(\bar{e}_j)$ is an orthogonal projection. □

At this point the fermionic character has entered the scene via (8) and the calculations become invalid for bosons. Lemma III.3 can be applied to general operators by dint of an operator version of Cauchy–Schwarz's inequality especially tailored to our needs. Its proof mimics one of the elementary proofs.

*Proposition III.4: Let $a_j, b_j:\mathcal{H}\to\mathcal{H}$ be bounded operators on a Hilbert space $\mathcal{H}$. Then, for $\sigma\in\{-1,1\}$,*

$$\sigma \sum_{j,k=1}^{M} a_j^* b_k^* b_j a_k \leq \sum_{j,k=1}^{M} a_j^* b_k^* b_k a_j.$$

*Proof:* Just look at the difference of both sides,

$$
\begin{aligned}
2\sum_{j,k}(a_j^*b_k^*b_ka_j-\sigma a_j^*b_k^*b_ja_k) &= 2\sum_{j,k}a_j^*b_k^*(b_ka_j-\sigma b_ja_k)\\
&= \sum_{j,k}a_j^*b_k^*(\sigma^2 b_ka_j-\sigma b_ja_k)+\sum_{j,k}a_k^*b_j^*(b_ja_k-\sigma b_ka_j)\\
&= \sum_{j,k}(\sigma a_j^*b_k^*(\sigma b_ka_j-b_ja_k)+a_k^*b_j^*(b_ja_k-\sigma b_ka_j))\\
&= \sum_{j,k}(\sigma a_j^*b_k^*-a_k^*b_j^*)(\sigma b_ka_j-b_ja_k)\\
&= \sum_{j,k}(\sigma b_ka_j-b_ja_k)^*(\sigma b_ka_j-b_ja_k)\\
&\geq 0.
\end{aligned}
$$

This implies the inequality. □

Now we can prove the first of the main theorems.

**Theorem III.5:** *Let $B\in B_r(L)$, $1\leq r\leq\infty$, and $s:=\frac{2(r-1)}{r}$. Then,*

$$
d\Gamma(B)^*d\Gamma(B)\leq\begin{cases}\|B\|_r^2N^s+\|B\|_2^2\mathbb{1} & 1<r<2\\ \|B\|_r^2N^s & r=1,\ \ 2\leq r\leq\infty.\end{cases}
$$

*Proof:* First of all, recall the singular value decomposition (19). The simplest case $r=1$ follows immediately from

$$
\left\|\sum_j \mu_j a^\dagger(f_j)a(\bar{e}_j)\right\|\leq\sum_j|\mu_j|=\|B\|_1.
$$

On to the other cases. By the Cauchy–Schwarz inequality III.4,

$$
\begin{aligned}
d\Gamma(B)^*d\Gamma(B) &= \sum_{j,k}a^\dagger(e_j)a(\overline{Be_j})a^\dagger(Be_k)a(\bar{e}_k)\\
&= -\sum_{j,k}a^\dagger(e_j)a^\dagger(Be_k)a(\overline{Be_j})a(\bar{e}_k)+\sum_{j,k}(Be_j,Be_k)a^\dagger(e_j)a(\bar{e}_k)\\
&\leq \sum_{j,k}\frac{\gamma_j^2}{\gamma_k^2}a^\dagger(e_j)a^\dagger(Be_k)a(\overline{Be_k})a(\bar{e}_j)+\sum_{j,k}(Be_j,Be_k)a^\dagger(e_j)a(\bar{e}_k)\\
&=: \Sigma_2+\Sigma_1,
\end{aligned}
$$

where $\gamma_j\in\mathbb{R}$, $\gamma_j\neq 0$, to be chosen appropriately.

Let $1<r<2$. By dint of (19) and Lemma III.3,

$$
\Sigma_2=\sum_k\frac{\mu_k^2}{\gamma_k^2}\sum_j\gamma_j^2a^\dagger(e_j)a(\bar{e}_j)\leq\sum_k\frac{\mu_k^2}{\gamma_k^2}\left(\sum_j\gamma_j^{2p}\right)^{1/p}N^{1/q},
$$

with $\frac{1}{p}+\frac{1}{q}=1$. Upon choosing $\gamma_k=\mu_k^\alpha$ we obtain

$$
\Sigma_2\leq\sum_k\mu_k^{2(1-\alpha)}\left(\sum_j\mu_j^{2\alpha p}\right)^{1/p}N^{1/q}.
$$

We want $2(1-\alpha)=r$ and $2\alpha p=r$ which implies

$$
\alpha=1-\frac{r}{2},\quad p=\frac{r}{2-r},
$$

with $1<p<\infty$. Then,

$$\Sigma_2 \le \left(\sum_j \mu_j^r\right)^{2/r} N^{2(r-1)/r}$$

after some calculations. The sum $\Sigma_1$ can be estimated by

$$\Sigma_1 = \sum_j \mu_j^2 a^\dagger(e_j)a(\bar{e}_j) \le \sum_j \mu_j^2 \mathbb{1} = \|B\|_2^2 \mathbb{1},$$

where the right-hand side is well-defined since $\|B\|_2 \le \|B\|_r$ for $1 \le r \le 2$.

For $2 \le r < \infty$ we put $\gamma_j = 1$ and use a different order of the factors in $\Sigma_2$,

$$\begin{aligned}\Sigma_2 &= \sum_{j,k} \mu_k^2 a^\dagger(f_k)a^\dagger(e_j)a(\bar{e}_j)a(\bar{f}_k)\\ &\le \sum_k \mu_k^2 a^\dagger(f_k)Na(\bar{f}_k)\\ &= \sum_k \mu_k^2 a^\dagger(f_k)a(\bar{f}_k)N - \sum_k \mu_k^2 a^\dagger(f_k)a(\bar{f}_k)\\ &= N^{1/2}\sum_k \mu_k^2 a^\dagger(f_k)a(\bar{f}_k)N^{1/2} - \Sigma_1,\end{aligned}$$

where we used that $N^{1/2}$ commutes with number preserving operators. By Lemma III.3,

$$\Sigma_2 \le \|B\|_r^2 N^{r-2/r+1} - \Sigma_1,$$

which proves the present case.

The case $r=\infty$ needs a bit more care since we do not avail of a singular value decomposition beforehand. Therefore, we look at the partial sums,

$$d\Gamma_M(B) = \sum_{j=1}^M a^\dagger(Be_j)a(\bar{e}_j).$$

The finite dimensional restriction,

$$B_M \coloneqq B|\mathrm{span}\{e_1, \ldots, e_M\},$$

however, does have a singular value decomposition, the singular values $\mu_j^{(M)}$ satisfying $\mu_j^{(M)} \le \|B\| = \|B^*\|$ by the min-max principle. Therefore, we can prove

$$\sum_{j,k=1}^M a^\dagger(e_j)a^\dagger(Be_k)a(\overline{Be_k})a(\bar{e}_j) \le \|B\|^2$$

$$\sum_{j=1}^M a^\dagger(e_j)Na(\bar{e}_j) \sum_{j,k=1}^M (Be_j,Be_k)a^\dagger(e_j)a(\bar{e}_k) \le \|B\|^2 \sum_{j=1}^M a^\dagger(e_j)a(\bar{e}_j).$$

Thus,

$$\begin{aligned}d\Gamma_M(B)^* d\Gamma_M(B) &\le \|B\|^2 \sum_{j=1}^M a^\dagger(e_j)Na(\bar{e}_j) + \|B\|^2 \sum_{j=1}^M a^\dagger(e_j)a(\bar{e}_j)\\ &= \|B\|^2 N^{1/2} \sum_{j=1}^M a^\dagger(e_j)a(\bar{e}_j)N^{1/2}\\ &\le \|B\|^2 N^2.\end{aligned}$$

That completes the proof. □

Now we turn to $\Delta(A)$ and $\Delta^+(C)$. Recall that $A$ and $C$ must be Hilbert–Schmidt operators for $\Delta(A)$ and $\Delta^+(C)$ to be well-defined whence the following theorems only make sense for $1 \le r \le 2$. Since Theorem III.5 contains the underlying ideas and computational details, we may be rather sketchy with the proofs.

**Theorem III.6:** *Let $A \in B_r^-(L)$, $1 \le r \le 2$, and $s := \frac{2(r-1)}{r}$. Then,*

$$\Delta(A)^*\Delta(A) \le \begin{cases} \|A\|_1^2 \mathbb{1} & r=1 \\ \|A\|_r^2 N^s + \|A\|_2^2 \mathbb{1} & 1 < r \le 2. \end{cases}$$

*Proof:* We use singular value decomposition (19). The case $r=1$ is obvious. For $1<r\le 2$ we start, as in Theorem III.5, from

$$\Delta(A)^*\Delta(A) = \sum_{j,k} \mu_j \mu_k a^\dagger(e_j) a^\dagger(\bar{f}_j) a(f_k) a(\bar{e}_k)$$
$$= -\sum_{j,k} \mu_j \mu_k a^\dagger(e_j) a(f_k) a^\dagger(\bar{f}_j) a(\bar{e}_k) + \sum_j \mu_j^2 a^\dagger(e_j) a(\bar{e}_j).$$

For $1<r<2$ the proof runs along the same lines as in Theorem III.5. However, for $r=2$ Cauchy–Schwarz's inequality III.4 gives us

$$\Delta(A)^*\Delta(A) \le \sum_{j,k} \mu_k^2 a^\dagger(e_j) a(f_k) a^\dagger(\bar{f}_k) a(e_j) + \sum_{j=1}^{M} \mu_j^2 a^\dagger(e_j) a(\bar{e}_j) \le \|A\|_2^2 N + \|A\|_2^2 \mathbb{1}.$$

That completes the proof. □

The remaining operator $\Delta^+(C)$ could be treated in like manner. However, it might be insightful to use an alternative idea. Note that generally an estimate for an operator does not yield an estimate for its adjoint.

**Theorem III.7:** *Let $C \in B_r^-(L)$, $1 \le r \le 2$, and $s := \frac{2(r-1)}{r}$. Then,*

$$\Delta^+(C)^*\Delta^+(C) \le \begin{cases} \|C\|_1^2 \mathbb{1} & r=1 \\ \|C\|_r^2 N^s + 3\|C\|_2^2 \mathbb{1} & 1 < r \le 2. \end{cases}$$

*Proof:* The case $r=1$ is obvious. For $1<r\le 2$ we use the commutator $[\Delta, \Delta^+]$ from (17) to obtain

$$\Delta^+(C)^*\Delta^+(C) = \Delta(C^*)\Delta^+(C)$$
$$= \Delta^+(C)\Delta(C^*) + [\Delta(C^*), \Delta^+(C)]$$
$$= \Delta(C^*)^*\Delta(C^*) - 4d\Gamma(CC^*) + 2 \operatorname{tr} C^*C \cdot \mathbb{1}.$$

Now use $d\Gamma(CC^*) \ge 0$ and Theorem III.6 to complete the proof. □

By using directly the defining series, one could obtain better estimates, e.g., for $r=2$,

$$\Delta^+(C)^*\Delta^+(C) \le \|C\|_2^2 (N + 2\mathbb{1}).$$

It is instructive to look at the bounds from the literature alluded to in Sec. I. Robinson[8] considered the special expression $\Delta^+(C)^k \Omega$ for $r=2$ $(s=1)$. Carey and Ruijsenaars[3] (2.14, 2.24, 2.25) have

$$d\Gamma(B)^* d\Gamma(B) \le \|B\|_\infty N^2, \tag{20}$$

$$\Delta(A)^*\Delta(A) \le \|A\|_2^2 N^2, \quad \Delta^+(C)^*\Delta^+(C) \le \|C\|_2^2(N+2\mathbb{1})^2. \tag{21}$$

When we assume $B$ just to be bounded, which is possible, then estimate (20) for $d\Gamma(B)$ is optimal. However, since $\Delta(A)$ and $\Delta^+(C)$ require $A$ and $C$ to be Hilbert–Schmidt operators rather than bounded operators (21) does not give the correct magnitude at all.

The estimates by Grosse and Langmann[5] [(70), Appendices B(b) and B(d)] are derived in a superversion of the CCR and CAR. Being valid for bosons and fermions alike they cannot reflect the special fermionic features used herein.

## IV. CONVERSE THEOREMS

Having seen Theorems III.5, III.6, and III.7, one would first and foremost ask whether the bounds given there are sharp. Since this is not really interesting for $\dim L<\infty$ we tacitly assume $\dim L=\infty$. We start with $d\Gamma(B)$ as this is the case which can be treated by elementary means. The following statement for $r=1$ is also mentioned, without proof, by Carey and Ruijsenaars[3] (p. 7).

**Theorem IV.1:** *Let $B\in B_\infty(L)$ and $d\Gamma(B)$ satisfy*

$$d\Gamma(B)^* d\Gamma(B) \le \gamma_r N^s + \delta_r \mathbb{1}, \quad s = \frac{2(r-1)}{r}, \quad 1 \le r < \infty. \tag{22}$$

*Then $B\in B_1(L)$ for $s=0$. When $0<s<2$ then $B\in B_{r+\varepsilon}(L)$ for all $\varepsilon>0$.*

*Proof:* Let $\{e_j\}$ be any ONS. We start with the formula

$$(a^\dagger(e_n)\cdots a^\dagger(e_1)\Omega, d\Gamma(B)a^\dagger(e_n)\cdots a^\dagger(e_1)\Omega) = \sum_{j=1}^n (e_j, Be_j), \tag{23}$$

which along with bound (22) implies

$$\left|\sum_{j=1}^n (e_j, Be_j)\right| \le (\gamma_r n^s + \delta_r)^{1/2}. \tag{24}$$

At first, we consider the special case of self-adjoint $B$. Then, either $(e_j,Be_j)\ge 0$ or $(e_j,Be_j)<0$. For the ONS at hand we may permute the indices as we wish without changing the right-hand side in (24). Hence, with some constant $\gamma$,

$$\sum_{j=1}^n |(e_j, Be_j)| \le \gamma n^{s/2}, \tag{25}$$

which in turn shows $(e_j,Be_j)\to 0$. If this were not so, there would be an $\varepsilon>0$, such that $|(e_j,Be_j)|\ge\varepsilon$ infinitely often. By the permutation argument this would contradict (25) since $0\le s<2$. Thus, we have shown that $(e_j,Be_j)\to 0$ for all ONS in $L$ which implies $B$ is compact (see, e.g., Bakić and Guljaš[1]). Using in (25) the ONS from singular value decomposition (19), we obtain

$$\sum_{j=1}^n \mu_j \le \gamma n^{s/2}, \tag{26}$$

where we noted $(e_j,f_j)=\pm 1$. For $s=0$ this implies $B\in B_1(L)$. Let $s>0$. From (26) we obtain the estimate

$$\mu_n \le n^{s/2-1}.$$

For the powers $\mu_n^r$ to be summable it suffices that $r\left(1-\frac{s}{2}\right)>1$. This is equivalent to $\frac{2(r-1)}{r}>s$ which implies the statement for self-adjoint $B$.

For general operators $B$ take real and imaginary parts in (23) and note $d\Gamma(B)^* = d\Gamma(B^*)$. Applying the first part to $B+B^*$ and $i(B-B^*)$ completes the proof. □

For the operators $\Delta(A)$ and $\Delta^+(C)$, we need more machinery, in particular, exponential functions of $\Delta^+(C)$. Fortunately, it is enough to define them on the vacuum,

$$\exp(z\Delta^+(C))\Omega, \quad z \in \mathbb{C},$$

where the exponential is defined via the power series. Such expressions were studied by Robinson[8] and called fermionic Gaussians. In physics one encounters the name BCS states. Their scalar product turns out to be an entire analytic function in $z$.

*Lemma IV.2: Let $C \in B_2^-(L)$. Assume*

$$\Delta^+(C)^*\Delta^+(C) \leq \gamma_r N^s + \delta_r \mathbb{1}, \quad s := \frac{2(r-1)}{r} \tag{27}$$

*for some $1 \leq r \leq 2$. Then, the function*

$$\omega(z) := (\exp(\bar{z}\Delta^+(C))\Omega, \exp(z\Delta^+(C)\Omega))$$

*is analytic on $\mathbb{C}$ and of exponential order $r$.*

*Proof:* Recall from (16) that $\Delta^+(C): \mathcal{F}^{(n)} \to \mathcal{F}^{(n+2)}$ and $\mathcal{F}^{(m)} \perp \mathcal{F}^{(n)}$ for $m \neq n$. Then,

$$\omega(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(n!)^2} (\Delta^+(C)^n\Omega, \Delta^+(C)^n\Omega).$$

Since the constants do not matter we may simplify the right-hand side of (27) to

$$\Delta^+(C)^*\Delta^+(C) \leq \gamma(N^s + \mathbb{1}),$$

with $s = \frac{2(r-1)}{r}$ and some appropriate $\gamma$. Unfortunately, such estimates do not transfer generally to powers of operators. Therefore, we have to estimate by hand

$$(\Delta^+(C)^{n+1})^*\Delta^+(C)^{n+1} \leq \gamma(\Delta^+(C)^n)^*(N^s + \mathbb{1})\Delta^+(C)^n.$$

We know $\Delta^+(C)^n\Omega \in \mathcal{F}^{(2n)}$ and $N|\mathcal{F}^{(2n)} = 2n\mathbb{1}|\mathcal{F}^{(2n)}$. Hence,

$$(\Omega, (\Delta^+(C)^{n+1})^*\Delta^+(C)^{n+1}\Omega) \leq \gamma((2n)^s + 1)(\Omega, (\Delta^+(C)^n)^*\Delta^+(C)^n\Omega).$$

Successively,

$$(\Omega, (\Delta^+(C)^{n+1})^*\Delta^+(C)^{n+1}\Omega) \leq \gamma^{n+1}((2n)^s + 1)((2(n-1))^s + 1)\cdots 1 \leq \gamma^{n+1}2^{(n+1)(s+1)}((n+1)!)^s,$$

where the last estimate is for convenience. With an appropriate $\tilde{z}$,

$$|\omega(z)| \leq \sum_{n=0}^{\infty} \frac{|z|^{2n}\gamma^n 2^{n(s+1)}}{(n!)^{2-s}} = \sum_{n=0}^{\infty} \frac{\tilde{z}^{2n}}{(n!)^{2/r}}$$

since $2 - s = \frac{2}{r}$. This shows $\omega$ is an entire function. Since $1 \leq r \leq 2$, we may use the classical Jensen inequality to deduce

$$|\omega(z)| \leq \sum_{n=0}^{\infty} \left(\frac{\tilde{z}^{nr}}{n!}\right)^{2/r} \leq \left(\sum_{n=0}^{\infty} \frac{\tilde{z}^{nr}}{n!}\right)^{2/r} = \exp\left(\frac{2}{r}\tilde{z}^r\right).$$

Hence, $\omega$ is of exponential order $r$. □

Lemma IV.2 pertains to Fock space properties of $\exp(z\Delta^+(C))$. On the other hand, we can express the scalar product on $\mathcal{F}$ through operators on $L$. See, e.g., Robinson.[8]

*Proposition IV.3: Let $C \in B_2^-(L)$ and $z \in \mathbb{C}$. Then,*

$$(\exp(\bar{z}\Delta^+(C))\Omega, \exp(z\Delta^+(C))\Omega) = \det(\mathbb{1} + 4z^2 C^* C).$$

Combining Lemma IV.2 with the determinant in Proposition IV.3, hopefully, will tell us something about $C$. To this end, we use a corollary of Jensen's integral formula from complex analysis that relates the distribution of zeros of entire functions with their exponential order. See Favorov[4] for the statement and some refinements.

**Theorem IV.4:** *Let $C \in B_2^-(L)$. If $\Delta^+(C)$ satisfies the estimate*

$$\Delta^+(C)^* \Delta^+(C) \leq \gamma_r N^s + \delta_r \mathbb{1}, \quad s = \frac{2(r-1)}{r}, \quad 1 \leq r \leq 2, \tag{28}$$

*then $C \in B_{r+\varepsilon}^-(L)$ for all $\varepsilon > 0$.*

*Proof:* We use the formula from Proposition IV.3,

$$\omega(z) := (\exp(\bar{z}\Delta^+(C))\Omega, \exp(z\Delta^+(C)\Omega)) = \det(\mathbb{1} + z^2 C^* C).$$

Lemma IV.2 and (28) imply $\omega$ has exponential order $r$. Because of Proposition IV.3 the zeros $z_j \neq 0$ of $\omega$ are given through the singular values $\mu_j$ of $C$,

$$z_j = \pm \frac{i}{\mu_j} \quad \text{for all} \quad \mu_j \neq 0.$$

The theorem from complex analysis mentioned above tells us

$$2 \sum_j \mu_j^\alpha = \sum_j \frac{1}{|z_j|^\alpha} < \infty$$

for all $\alpha > r$. Hence, $C \in B_\alpha^-(L)$ for all $\alpha > r$. □

Theorem IV.4 can be used for $\Delta(A)$ by the same reasoning as in Theorem III.7.

**Theorem IV.5:** *Let $A \in B_2^-(L)$. If $\Delta(A)$ satisfies the estimate*

$$\Delta(A)^* \Delta(A) \leq \gamma_r N^s + \delta_r \mathbb{1}, \quad s = \frac{2(r-1)}{r}, \quad 1 \leq r \leq 2, \tag{29}$$

*then $A \in B_{r+\varepsilon}^-(L)$ for all $\varepsilon > 0$.*

*Proof:* As in Theorem III.7 we obtain the estimate

$$\Delta^+(A^*)^* \Delta^+(A^*) \leq \gamma_r N^s + \delta_r \mathbb{1} + 2 \operatorname{tr} A^* A \cdot \mathbb{1}.$$

Then, Theorem IV.4 yields the statement. □

Theorems IV.1, IV.5, and IV.4 naturally make one come up with the question as to whether the $\varepsilon$ could be removed there. Except for one special case, $r=1$ in Theorem III.5, this is an open problem. If we could get rid of $\varepsilon$ the bounds in Sec. IIIwould become sharp, at least asymptotically. That this is so was conjectured by Carey and Ruijsenaars[3] and Ottesen[7] for the case $r=1$. Our proofs as they stand cannot be generalized. The estimate of the singular values in Theorem IV.1 is sharp as show simple examples. As to Theorem IV.4 there are entire functions of exponential order 1 whose zeros cannot be summed up with exponent 1, e.g., $f(z)=\sin(z)$. Hence, although the operators

$$d\Gamma(B) = \sum_j \frac{1}{j} a^\dagger(f_j) a(\bar{e}_j), \quad \Delta^+(C) = \sum_j \frac{1}{j} a^\dagger(f_j) a^\dagger(\bar{e}_j)$$

look quite similar, we only know the first to be unbounded whereas the latter's unboundedness remains an open problem.

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