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An adiabatic theorem for section determinants of spectral projections [1]

By PETER OTTE of Bochum

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Abstract. We derive an adiabatic-type theorem that expresses the section determinants of spectral projections of a self-adjoint operator through the solution to an operator-valued Wiener-Hopf equation. The solution theory of this equation is developed and for a special case a concrete criterion that ensures uniqueness of the solution is presented. Furthermore, for a special class of operators a dichotomy criterion, which is used in the proof of the adiabatic theorem, is proved.

1. Introduction

The classical adiabatic theorems relate the spectral projections of self-adjoint operators to the solutions to time-dependent Schrödinger equations (see e.g. [3]). This relationship is established by a special limiting process, which can be interpreted as though a perturbation is being switched on infinitely slowly or, in physics language, adiabatically. Here we present a new adiabatic theorem that expresses the section determinants of the spectral projections by means of the solution to an operator-valued Wiener-Hopf equation. More precisely, let A_0 be a self-adjoint and B be a bounded self-adjoint operator acting on a Hilbert space $\mathcal{H}$. Let $P(0)$ and $P(1)$ be the spectral projections to certain parts of the spectrum of A_0 and $A_0 + B$, respectively. We assume $\dim \operatorname{ran} P(0) = N$ with $1 \leq N < \infty$ and consider the section determinant $\det(P(0)P(1)P(0))$, which is to be understood as the determinant of the finite dimensional operator $P(0)P(1)P(0) : \operatorname{ran} P(0) \to \operatorname{ran} P(0)$. Such determinants occur in quantum mechanics where they describe transition probabilities within a system consisting of N fermions that is exposed to a sudden perturbation. In Theorem 4.17, we prove the formula:

(1.1)
$$\det(P(0)P(1)P(0)) = \lim_{\eta\to+0}\lim_{\varepsilon\to+0} \exp\left[-2\operatorname{Re}\int_0^1\int_{-\infty}^0 e^{\eta t}\operatorname{tr} BG_\varepsilon(t,t+0;\alpha)\,dt\,d\alpha\right],$$

[1] This paper is an improved version of parts of the author's PhD thesis.

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which may be considered a variant of the adiabatic theorem. The operator $G_\varepsilon(t,t';\alpha)$ solves the Wiener-Hopf equation:

(1.2)
$$G_\varepsilon(t,t';\alpha)\varphi = e^{-\varepsilon|t-t'|}G_0(t-t')\varphi + \alpha\int_{-\infty}^0 e^{-\varepsilon|t-\tau|}G_0(t-\tau)BG_\varepsilon(\tau,t';\alpha)\varphi\, d\tau,\ \varphi\in\mathcal{H},$$

where the operator $G_0(t)$ can be expressed in terms of A_0 and $P(0)$. The operator $G_\varepsilon(t,t';\alpha)$ is called time-ordered Green operator although other names appear (see [4]). In the physics literature, the solution to Schrödinger equations with time-dependent potential is written as a formal exponential series, the so-called time-ordered exponential function. This may explain the name time-ordered Green operator.

The parameters η and ε are related to certain gap conditions to be imposed on the operator A. The first one requires a gap in the spectrum of A exactly as in the classical case whereas the parameter ε is related to a gap in the numerical range of A. This is a new feature of our adiabatic theorem and unknown to the old one. As we shall see in Sec. 3 this gap condition ensures that a certain operator has no real spectral values, which is essential to derive the Wiener-Hopf equation. Such dichotomous operators have been investigated before, see e.g. [5] or [7]. However, the criteria derived as yet do not meet our needs.

Wiener-Hopf equations as in (1.2) were formerly studied by Bart, Gohberg, and Kaashoek [4] (for the general theory see e.g. [11]). We shall not use their results here but develop independently a solution theory for the homogeneous equation in Sec. 5. Moreover, in Sec. 6 we present a concrete uniqueness theorem that is formulated solely in terms of the operator A and thus avoids referring to the section determinant of the spectral projections.

Leaving aside the limits, Formula (1.1) bears some similarity to a formula for section determinants of operators that are solutions to Schrödinger equations (see [8]). Indeed, one might think of using a generalization of this formula along with the classical adiabatic theorem (see [3]) to provide an alternative approach to (1.1) as is discussed in [9]. This idea was employed for the first time by Rivier and Simanek [13] on a purely formal basis to compute the exact asymptotics in Anderson's orthogonality theorem [1]. Although we think that the proof given here is more instructive and easier than making this approach rigorous, this connection should justify the name adiabatic theorem for (1.1).

2. Preliminaries

The algebraic part of the adiabatic theorem is based upon a well-known formula that generalizes the logarithmic derivative of a scalar-valued function to the matrix case.

Lemma 2.1. *Let $\mathcal{K}$ be a finite dimensional Hilbert space and $S(\alpha):\mathcal{K}\to\mathcal{K}$, $\alpha_0\le\alpha\le\alpha_1$, be a family of invertible linear operators depending continuously differentiably on α in the operator norm. Then,*

$$\frac{d}{d\alpha}\ln\det S(\alpha) = \operatorname{tr}(S^{-1}(\alpha)S'(\alpha)). \tag{2.1}$$

Here a prime denotes the derivative with respect to α *and* tr *is the trace.*

Proof. Let $s_1(\alpha), \dots, s_N(\alpha)$ denote the column vectors of $S(\alpha)$ where $N := \dim \mathcal{K}$. Then,

$$\frac{d}{d\alpha} \det S(\alpha) = \sum_{k=1}^{N} \det(s_1(\alpha), \dots, s_{k-1}(\alpha), s_k'(\alpha), s_{k+1}(\alpha), \dots, s_N(\alpha)).$$

Expand the kth summand by the kth column and recall Cramer's rule to infer that

$$\frac{d}{d\alpha} \det S(\alpha) = \det S(\alpha) \operatorname{tr}(S^{-1}(\alpha) S'(\alpha)).$$

This proves the lemma. □

The trace in Lemma 2.1 is the reason why we shall need some simple facts about trace class operators. We generally refer to [12] for the necessary prerequisites. It is easily seen by standard arguments that $\operatorname{tr}(S^{-1}(\alpha)S'(\alpha))$ depends continuously on α whence we may integrate the above formula. Exponentiating yields:

$$\det S(\alpha_1) = \det S(\alpha_0) \exp \left[\int_{\alpha_0}^{\alpha_1} \operatorname{tr} S^{-1}(\alpha) S'(\alpha) \, d\alpha \right]. \tag{2.2}$$

In order to apply Formula (2.2) to section determinants we need to extend the inverse $(PP(\alpha)P)^{-1}$ from $\operatorname{ran} P$ to $\mathcal{H}$. We do this in a trivial way by first defining

$$S^+ \varphi := \begin{cases} (PSP|_{\operatorname{ran} P})^{-1} \varphi & \text{for } \varphi \in \operatorname{ran} P, \\ 0 & \text{for } \varphi \in (\operatorname{ran} P)^\perp \end{cases} \tag{2.3}$$

and then extending S^+ by linearity to the whole of $\mathcal{H}$. We call S^+ pseudo inverse of S relative to P.

Lemma 2.2. *Let* $S : \mathcal{H} \to \mathcal{H}$ *be a linear operator and* $P : \mathcal{H} \to \mathcal{H}$ *be an orthogonal projection such that the operator* $(PSP|_{\operatorname{ran} P})^{-1} : \operatorname{ran} P \to \operatorname{ran} P$ *exists. Then the pseudo inverse* S^+ *of* S *relative to* P *is well-defined and has the properties:* $S^+ = S^+P = PS^+$ *and* $S^+SP = PSS^+ = P$. *If in addition* S *and* P *commute, we have* $SS^+ = S^+S = P$. *If* S *is self-adjoint so is* S^+.

Proof. Straightforward linear algebra (see [8]). □

We rewrite the determinant formula in Lemma 2.1 for orthogonal projections. The need for doing so will become clear in Lemma 4.3 below.

Proposition 2.3. *Let* $P(\alpha) : \mathcal{H} \to \mathcal{H}$, $0 \le \alpha \le 1$, *be a family of orthogonal projections depending continuously differentiably in the operator norm on* α. *Let further* $P : \mathcal{H} \to \mathcal{H}$ *be an orthogonal projection having finite dimensional range such that* $\det(PP(\alpha)P) \neq 0$ *for all* $\alpha \in [0,1]$. *Then,*

$$\det(PP(1)P) = \det(PP(0)P) \exp \left[2 \operatorname{Re} \int_0^1 \operatorname{tr} P^+(\alpha)(\mathbb{1} - P(\alpha)) P'(\alpha) P(\alpha) \, d\alpha \right]. \tag{2.4}$$

Here $P^+(\alpha)$ is the pseudo inverse of $P(\alpha)$ relative to P according to (2.3) *and* $\mathbb{1} : \mathcal{H} \to \mathcal{H}$ *is the identity operator.*

Proof. Since $P(\alpha)$ depends continuously differentiably on α and $\det(PP(\alpha)P) \neq 0$ we can apply Formula (2.2) to the finite dimensional operator $PP(\alpha)P : \operatorname{ran} P \to \operatorname{ran} P$:

$$\det(PP(1)P) = \det(PP(0)P) \exp\left[\int_0^1 \operatorname{tr} P^+(\alpha)P'(\alpha)\, d\alpha\right].$$

Here we noted $P^+(\alpha) = PP^+(\alpha)$ (see Lemma 2.2). Differentiating the identity $P(\alpha) = P(\alpha)^2$ yields:

$$P'(\alpha) = P(\alpha)P'(\alpha) + P'(\alpha)P(\alpha). \tag{2.5}$$

$P^+(\alpha)$ and $P'(\alpha)$ are self-adjoint because $P(\alpha)$ is. We note $\operatorname{tr} S^* = (\operatorname{tr} S)^*$ for any linear operator S. Then,

$$\begin{aligned}
\operatorname{tr} P^+(\alpha)P'(\alpha) &= \operatorname{tr} P^+(\alpha)[P(\alpha)P'(\alpha) + P'(\alpha)P(\alpha)] \\
&= (\operatorname{tr} P^+(\alpha)P'(\alpha)P(\alpha))^* + \operatorname{tr} P^+(\alpha)P'(\alpha)P(\alpha) \\
&= 2 \operatorname{Re} \operatorname{tr} P^+(\alpha)P'(\alpha)P(\alpha).
\end{aligned}$$

Multiplying (2.5) by $P(\alpha)$ shows $P(\alpha)P'(\alpha)P(\alpha) = 0$. Hence,

$$P'(\alpha)P(\alpha) = (\mathbb{1} - P(\alpha))P'(\alpha)P(\alpha).$$

This completes the proof. □

Thus far we described the more algebraic ingredients of the adiabatic theorem. Now we turn to the analysis part. The starting point is a formula due to Riesz for the spectral projections in the special case that the spectrum is decomposable into two disjoint closed sets. Let $S : D(S) \to \mathcal{H}$ be a closed, densely defined, linear operator. We denote by $\rho(S)$ and $\sigma(S)$ the resolvent set and spectrum of S, respectively. Let $\sigma_1(S), \sigma_2(S) \subset \sigma(S)$ be closed sets with $\sigma_1(S) \cap \sigma_2(S) = \emptyset$ and $\sigma_1(S) \cup \sigma_2(S) = \sigma(S)$. Such sets are called spectral sets of S. We can choose a closed, sufficiently smooth, and positively oriented path Γ in the complex plane such that $\sigma_1(S)$ is enclosed by Γ and $\sigma_2(S)$ lies outside Γ. We call $\sigma_1(S)$ the spectral set of S with respect to Γ. The spectral projection associated with $\sigma_1(S)$ is given by Riesz' formula:

$$P = \frac{1}{2\pi i} \int_\Gamma (z\mathbb{1} - S)^{-1}\, dz. \tag{2.6}$$

Since the resolvent depends continuously on z in the operator norm the integral is defined as a Riemann integral in the operator norm. In the sequel, all integrals involving only the resolvent are to be understood in this sense without further mention. P commutes with S, i.e. $P\varphi \in D(S)$ for all $\varphi \in \mathcal{H}$ and $SP\varphi = PS\varphi$ for all $\varphi \in D(S)$. Moreover, the operator $PSP : \operatorname{ran} P \to \operatorname{ran} P$ has $\sigma_1(S)$ as its spectrum and the restriction of the resolvent equals the resolvent of the restriction. Analogous statements hold for $\operatorname{ran}(\mathbb{1} - P)$. Further details and proofs may be found in [6].

We shall connect the spectral projections to a Wiener-Hopf equation via certain one-parameter families of operators $U(t) : \mathcal{H} \to \mathcal{H}$, $t \in \mathbb{R}$, the so-called strongly continuous groups of bounded linear operators, hereafter simply called groups. We refer to [10] for the necessary background. A group of operators satisfies the estimate

$$\|U(t)\| \leq M e^{\omega|t|}, \ t \in \mathbb{R}, \tag{2.7}$$

with constants $M \geq 1$ and $\omega \geq 0$. When $\sigma(S)$, S being the generator of the group, is decomposed into spectral sets $\sigma_1(S)$ and $\sigma_2(S)$ then $U(t)$ can be expressed by a Dunford-Taylor integral

$$U(t)P = \frac{1}{2\pi i} \int_\Gamma e^{zt}(z\mathbb{1} - S)^{-1}\, dz. \tag{2.8}$$

Here Γ is a path such that $\sigma_1(S)$ is the spectral set with respect to Γ and P is the associated spectral projection. This formula can be verified by checking that both sides solve the same initial value problem. In the literature the integration path Γ is often chosen to enclose the entire spectrum of S thus avoiding the projection P. Formula (2.8) has a converse:

$$(z\mathbb{1} - S)^{-1}\varphi = \int_{-\infty}^{0} e^{zt} U(-t)\varphi\, dt. \tag{2.9}$$

The integral is to be understood as a Riemann integral in the strong sense. In the sequel, the appearance of a vector $\varphi \in \mathcal{H}$ in an integral always indicates that the integral is a strong one in contrast to the integrals in the operator norm. Due to reasons stemming from physics in the present context the generator S will always look like $S = -i\tilde{S}$ with some operator $\tilde{S}$, which causes slight modifications of Formulae (2.8) and (2.9) above. This has to be taken into account in the respective calculations.

3. Operators with dichotomy

A closed linear operator $S : D(S) \to \mathcal{H}$, $D(S) \subset \mathcal{H}$, is dichotomous or has a dichotomy if $\sigma(S) \cap \mathbb{R} = \emptyset$. Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator. Surely, the operator $A + i\varepsilon\mathbb{1}$ is dichotomous if $\varepsilon \neq 0$ (see e.g. [12]). For the proof of the adiabatic theorem in Sec. 4, we need slightly more general imaginary parts. Given an orthogonal projection $P : \mathcal{H} \to \mathcal{H}$ we ask whether

$$A_\varepsilon := A + i\varepsilon(2P - \mathbb{1}), \ \varepsilon \in \mathbb{R} \setminus \{0\} \tag{3.1}$$

is dichotomous, which is answered by the theorem below. The key idea is to consider the Schur complement of $A_\varepsilon - x\mathbb{1}$, $x \in \mathbb{R}$, in $\operatorname{ran} P$. Related problems were dealt with in greater generality in [2]. However, the results therein rely on some compactness properties of the operators involved, which are not directly satisfied in our context, wherefore we prefer to present a complete proof here.

Theorem 3.1. *Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator and $P : \mathcal{H} \to \mathcal{H}$ be an orthogonal projection having a finite dimensional range and the property $P\varphi \in D(A)$ for all $\varphi \in \mathcal{H}$. We write $D_1 := \mathcal{H}_1 := \operatorname{ran} P$, $\mathcal{H}_2 := \mathcal{H}_1^\perp$, and $D_2 := D(A) \cap \mathcal{H}_2$.*

Let D_2 be dense in $\mathcal{H}_2$ and the operator $(\mathbb{1}-P)A(\mathbb{1}-P) : D_2 \to \mathcal{H}_2$ be self-adjoint. Finally, let $\lambda < \mu$ and assume

$$(\varphi, A\varphi) \le \lambda \|\varphi\|^2 \textit{ for all } \varphi \in D_1 \textit{ and } \mu\|\varphi\|^2 \le (\varphi, A\varphi) \textit{ for all } \varphi \in D_2. \tag{3.2}$$

Then, $A_\varepsilon := A + i\varepsilon(2P - \mathbb{1})$ is dichotomous for all $\varepsilon \in \mathbb{R} \setminus \{0\}$.

Proof. We write A_ε, $\varepsilon \in \mathbb{R} \setminus \{0\}$, as block matrix:

$$A_\varepsilon = \begin{pmatrix} A_{11} + i\varepsilon\mathbb{1} & A_{12} \\ A_{21} & A_{22} - i\varepsilon\mathbb{1} \end{pmatrix}, \; A_{jk} : D_k \to \mathcal{H}_j, \; j,k = 1,2,$$

where we do not distinguish between the different identity operators acting on different subspaces. By assumption $A_{22} : D_2 \to \mathcal{H}_2$ is self-adjoint whence the resolvent $R := (A_{22} - (i\varepsilon + x)\mathbb{1})^{-1}$ exists and is bounded. Therefore, the Schur complement S of $A_\varepsilon - x\mathbb{1}$ in $\mathcal{H}_1$ exists:

$$S := A_{11} - x\mathbb{1} + i\varepsilon\mathbb{1} - A_{12}RA_{21}, \; S : \mathcal{H}_1 \to \mathcal{H}_1.$$

We show that S is invertible. Since $\mathcal{H}_1$ is finite dimensional injectivity is enough. Let $\varphi \in \mathcal{H}_1$ with $S\varphi = 0$. We split the quadratic form $(\varphi, S\varphi)$ into real and imaginary part where we note $A_{12}^* = A_{21}$:

$$\begin{aligned} 0 &= (\varphi, S\varphi) \\ &= (\varphi, (A_{11} - x\mathbb{1})\varphi) - (RA_{21}\varphi, (A_{22} - x\mathbb{1})RA_{21}\varphi) + i\varepsilon\|\varphi\|^2 - i\varepsilon\|RA_{21}\varphi\|^2. \end{aligned}$$

Taking the real part and recalling the properties of λ and μ gives:

$$\begin{aligned} 0 &= (\varphi, (A_{11} - x\mathbb{1})\varphi) - (RA_{21}\varphi, (A_{22} - x\mathbb{1})RA_{21}\varphi) \\ &\le (\lambda - x)\|\varphi\|^2 - (\mu - x)\|RA_{21}\varphi\|^2. \end{aligned}$$

The imaginary part can be used to rewrite the second term:

$$0 \le (\lambda - x)\|\varphi\|^2 - (\mu - x)\|\varphi\|^2 = (\lambda - \mu)\|\varphi\|^2.$$

Since $\lambda - \mu < 0$ this is only possible with $\|\varphi\| = 0$, i.e. $\varphi = 0$.

Having S^{-1} at our disposal we can satisfy ourselves that the algebraic inverse of $A_\varepsilon - x\mathbb{1}$ is given by

$$(A_\varepsilon - x\mathbb{1})^{-1} = \begin{pmatrix} S^{-1} & S^{-1}A_{12}R \\ -RA_{21}S^{-1} & RA_{21}S^{-1}A_{12}R + R \end{pmatrix}.$$

Now we are left with proving that $(A_\varepsilon - x\mathbb{1})^{-1}$ is bounded, which is equivalent to all of its entries being bounded. This is clear for S^{-1} and A_{21} because they act on the finite dimensional subspace $\mathcal{H}_1$. Also R is bounded as we noted above. Moreover, it maps $\mathcal{H}_2$ to D_2, which implies by the closed graph theorem the boundedness of $A_{12}R$ wherewith the theorem is proven. □

It is easily seen that (3.2) may be replaced by

$$(\varphi, A\varphi) \ge -\lambda\|\varphi\|^2 \text{ for all } \varphi \in D_1 \text{ and } -\mu\|\varphi\|^2 \ge (\varphi, A\varphi) \text{ for all } \varphi \in D_2. \tag{3.3}$$

It is remarkable that Theorem 3.1 neither depends on the size of the off-diagonal elements nor on the size of the 'gap' between A_{11} and A_{22} as long as it differs from zero. Unlike this the validity of Baskakov's criterion [5] would entail, roughly speaking, that the real part is smaller than the imaginary part, which makes it inapplicable to our situation because of ε tending to zero. In [7] Langer and Tretter studied the absence of purely imaginary spectral values using gap conditions that perfectly resemble those we used. However, the appearance of the imaginary axis instead of the real axis in their results is crucial and, thus, impedes to employ their methods for the operators needed here.

4. The adiabatic theorem

We fix the notations and assumptions that will be used throughout this section. Let $A_0 : D(A_0) \to \mathcal{H}$ be a self-adjoint operator with domain $D_0 := D(A_0) \subset \mathcal{H}$. Further let $B : \mathcal{H} \to \mathcal{H}$ be a bounded self-adjoint operator. We seek a formula that relates the spectral projections of A_0 to those of $A_0 + B$. Since we want to apply Proposition 2.3 we need to differentiate somehow. Therefore, we introduce a new parameter α:

$$A(\alpha) := A_0 + \alpha B, \ \ \alpha \in [0,1]. \tag{4.1}$$

$A(\alpha)$ is self-adjoint on D_0 for all $\alpha \in [0,1]$ and thus $-iA(\alpha)$ generates a group of unitary operators $U(t,\alpha)$. For the proof of the adiabatic theorem, the operator pencil $A(\alpha)$ has to be subject to some conditions.

(A1) There is a path $\Gamma \subset \mathbb{C}$ such that $\Gamma \subset \rho(A(\alpha))$ for all $\alpha \in [0,1]$.

Setting $R(z,\alpha) := (z\mathbb{1} - A(\alpha))^{-1}$ this implies along with the boundedness of B that the function $(z,\alpha) \mapsto \|R(z,\alpha)\|$ is continuous on the compact set $\Gamma \times [0,1]$ and attains its maximum thereon:

$$m := \max_{(z,\alpha)\in\Gamma\times[0,1]} \|R(z,\alpha)\| < \infty. \tag{4.2}$$

We denote by $\sigma_1(A(\alpha))$ the spectral set of $A(\alpha)$ with respect to Γ and put $\sigma_2(A(\alpha)) := \sigma(A(\alpha)) \setminus \sigma_1(A(\alpha))$. Henceforth, the spectral sets $\sigma_1(A(\alpha))$ and $\sigma_2(A(\alpha))$ will always refer to this path Γ. Condition (A1) ensures that the spectral projections associated with $\sigma_1(A(\alpha))$ can be expressed with the aid of the same path Γ:

$$P(\alpha) = \frac{1}{2\pi i}\int_\Gamma R(z,\alpha)\,dz. \tag{4.3}$$

Each $P(\alpha)$ is an orthogonal projection and commutes with $U(t,\alpha)$. Moreover, $P(\alpha)$ depends continuously differentiably on α in the operator norm and the derivative is given by the Friedrichs integral:

$$P'(\alpha) = \frac{1}{2\pi i}\int_\Gamma R(z,\alpha)BR(z,\alpha)\,dz. \tag{4.4}$$

Additionally, we require $P(\alpha)$ to satisfy the following conditions:

(A2) $\dim \operatorname{ran} P(0) = N$ with $1 \le N < \infty$.

(A3) $\det(P(0)P(\alpha)P(0)) \neq 0$ for all $\alpha \in [0,1]$.

Condition (A2) says that $\sigma_1(A(0))$ contains exactly N eigenvalues counted with multiplicity. This is not necessary at all points. However, it is crucial in order that the determinant in (A3) is well-defined. (A3) implies the existence of the pseudo inverse of $P(\alpha)$ relative to $P(0)$.

Since $P(0)$ is a spectral projection of A_0 we have $P(0)\varphi \in D_0$ for all $\varphi \in \mathcal{H}$. This implies $P(0)\varphi \in D(A(\alpha)) = D_0$ since B is bounded. We set $D_1 := \operatorname{ran} P(0)$ and $D_2 := D_0 \cap D_1^{\perp}$.

(A4) For $\alpha \in [0,1]$ let $\lambda(\alpha) < \mu(\alpha)$. We assume exactly one of the following conditions to be satisfied:

1. $(\varphi, A(\alpha)\varphi) \le \lambda(\alpha)\|\varphi\|^2$ for $\varphi \in D_1$ and $\mu(\alpha)\|\varphi\|^2 \le (\varphi, A(\alpha)\varphi)$ for $\varphi \in D_2$.
2. $(\varphi, A(\alpha)\varphi) \ge -\lambda(\alpha)\|\varphi\|^2$ for $\varphi \in D_1$ and $-\mu(\alpha)\|\varphi\|^2 \ge (\varphi, A(\alpha)\varphi)$ for $\varphi \in D_2$.

This gap condition is exactly the dichotomy criterion 3.1. With (A4), we have completed the list of conditions to be imposed on $A(\alpha)$.

Differentiability of $P(\alpha)$ along with (A2) implies

$$\dim \operatorname{ran} P(\alpha) = N \text{ for all } \alpha \in [0,1], \tag{4.5}$$

which tells us that $\sigma_1(A(\alpha))$ contains exactly N eigenvalues counted with multiplicity and no other spectral values. The first step toward Theorem 4.17 consists in a simple application of Proposition 2.3 to $P(\alpha)$.

Proposition 4.1. *For the family of spectral projections $P(\alpha)$ of $A(\alpha)$ there holds:*

$$\det(P(0)P(1)P(0)) = \exp\left[2\operatorname{Re}\int_0^1 \operatorname{tr} P^+(\alpha)(\mathbb{1} - P(\alpha))P'(\alpha)P(\alpha)\,d\alpha\right]. \tag{4.6}$$

Here $P^+(\alpha)$ is the pseudo inverse of $P(\alpha)$ relative to $P(0)$ and $P'(\alpha)$ is given by the Friedrichs integral (4.4).

Proof. Each $P(\alpha)$ is an orthogonal projection. Because of the differentiability and (A3) the assumptions of Proposition 2.3 are satisfied. The derivative $P'(\alpha)$ is given by (4.4) and $\det(P(0)P(0)P(0)) = 1$ because of (A2). ☐

In the rest of this section we shall be faced with the task of casting the exponent in (4.6) into a form that can be characterized through known quantities. A purely formal application of Parseval's identity from Fourier theory to the Friedrichs integral in (4.6) would link this formula to the group generated by $-iA(\alpha)$. In order to make this idea rigorous we approximate $P'(\alpha)$ by

$$\Phi_\eta(\alpha) := \frac{1}{2\pi i}\int_\Gamma R(z + i\frac{\eta}{2}, \alpha)BR(z - i\frac{\eta}{2}, \alpha)\,dz,\ \eta \in \mathbb{R}. \tag{4.7}$$

In some sense we shifted the spectrum of $A(\alpha)$ into the complex plane. The factor $1/2$ has been introduced in order to yield $e^{\eta t}$ instead of $e^{2\eta t}$ in the adiabatic theorem.

Lemma 4.2. *Let $\Phi_\eta(\alpha)$ be as in (4.7). Then there is an $\eta_0 > 0$ such that $\Phi_\eta(\alpha)$ is well-defined for all η with $|\eta| \le \eta_0$ and all $\alpha \in [0,1]$. Moreover, $\lim_{\eta\to 0} \Phi_\eta(\alpha) = P'(\alpha)$ in the operator norm uniformly for $\alpha \in [0,1]$.*

Proof. Standard arguments involving the Neumann series. □

The parameter η enables us to link the Friedrichs integral $\Phi_\eta(\alpha)$ to the group $U(t,\alpha)$ via the Laplace transform.

Lemma 4.3. *Let $\Phi_\eta(\alpha)$ be as in (4.7). Then, for $\eta > 0$ small enough and all $\varphi \in \mathcal{H}$:*

$$(\mathbb{1} - P(\alpha))\Phi_\eta(\alpha)P(\alpha)\varphi = -i\int_{-\infty}^{0} e^{\eta t} U(-t,\alpha)(\mathbb{1} - P(\alpha))BU(t,\alpha)P(\alpha)\varphi\, dt \tag{4.8}$$

where $U(t,\alpha)$ is the group generated by $-iA(\alpha)$.

Proof. Let $\eta > 0$. Then in (4.7) Γ encloses only a finite number of poles of

$$(\mathbb{1} - P(\alpha))R(z + \frac{i\eta}{2},\alpha)BR(z - \frac{i\eta}{2},\alpha)P(\alpha)$$

and all of these poles lie in the upper half-plane. Cauchy's theorem allows us to replace Γ by Γ_+ with $\operatorname{Im} z \ge 0$ for $z \in \Gamma_+$:

$$(\mathbb{1} - P(\alpha))\Phi_\eta(\alpha)P(\alpha) = \frac{1}{2\pi i}\int_{\Gamma_+} (\mathbb{1} - P(\alpha))R(z + i\frac{\eta}{2},\alpha)BR(z - i\frac{\eta}{2},\alpha)P(\alpha)\, dz. \tag{4.9}$$

Since $\operatorname{Im}(z + \frac{i\eta}{2}) > 0$ for $z \in \Gamma_+$ and $\eta > 0$ we infer from (2.9):

$$R(z + i\frac{\eta}{2},\alpha)\varphi = -i(-i(z + i\frac{\eta}{2})\mathbb{1} - (-iA(\alpha)))^{-1} = -i\int_{-\infty}^{0} e^{-i(z + i\frac{\eta}{2})t} U(-t,\alpha)\varphi\, dt.$$

Inserting this into (4.9) yields:

$$\begin{aligned}
&(\mathbb{1} - P(\alpha))\Phi_\eta(\alpha)P(\alpha)\varphi \\
&= -\frac{1}{2\pi}\int_{\Gamma_+}\int_{-\infty}^{0} e^{-i(z + i\frac{\eta}{2})t} U(-t,\alpha)(\mathbb{1} - P(\alpha))BR(z - i\frac{\eta}{2},\alpha)P(\alpha)\varphi\, dt\, dz \\
&= -\frac{1}{2\pi}\int_{-\infty}^{0} e^{\frac{\eta}{2}t} U(-t,\alpha)(\mathbb{1} - P(\alpha))B\int_{\Gamma_+} e^{-izt} R(z - i\frac{\eta}{2},\alpha)P(\alpha)\, dz\, \varphi\, dt.
\end{aligned}$$

Interchanging the integrals is allowed because all integrands are absolutely integrable. To evaluate the inner integral we change variables $z = iw + i\frac{\eta}{2}$ and then use (2.8) noting

$$R(iw,\alpha) = \frac{1}{i}(w\mathbb{1} - (-i)A(\alpha))^{-1}.$$

This completes the proof. □

The variant of the determinant formula (2.2) for orthogonal projections in Lemma 2.3 enables us to handle the singularities of the resolvent in the above Lemma 4.3.

Proposition 4.4. *Let $P(\alpha)$ be the spectral projections of $A(\alpha)$ according to* (4.3). *Then,*

$$\begin{aligned} (4.10) \quad & \det(P(0)P(1)P(0)) \\ & = \lim_{\eta\to+0} \exp\left[-\int_0^1 2\operatorname{Re} i\int_{-\infty}^0 e^{\eta t}\operatorname{tr}(BU(t,\alpha)P(\alpha)P^+(\alpha)U(-t,\alpha))\,dt\,d\alpha\right]. \end{aligned}$$

Proof. We want to rewrite the exponent in Proposition 4.1. Because of the appearance of $P^+(\alpha)$, the trace refers only to the finite dimensional subspace $\operatorname{ran} P(0)$. Hence Lemma 4.2, which states convergence in the operator norm, implies:

$$\operatorname{tr}(P^+(\alpha)(\mathbb{1}-P(\alpha))P'(\alpha)P(\alpha)) = \lim_{\eta\to 0}\operatorname{tr}(P^+(\alpha)(\mathbb{1}-P(\alpha))\Phi_\eta(\alpha)P(\alpha)).$$

For $\eta > 0$ we may use the expression for $\Phi_\eta(\alpha)$ from Lemma 4.3:

$$\det(P(0)P(1)P(0)) =$$
$$\exp\left[-2\int_0^1 \lim_{\eta\to+0}\operatorname{Re} i\int_{-\infty}^0 e^{\eta t}\operatorname{tr}(P^+(\alpha)U(-t,\alpha)(\mathbb{1}-P(\alpha))BU(t,\alpha)P(\alpha))\,dt\,d\alpha\right].$$

We note $\operatorname{Re}(i\operatorname{tr} P^+(\alpha)U(-t,\alpha)P(\alpha)BP(\alpha)U(t,\alpha)) = 0$ since the trace of the product of two self-adjoint operators is real and use the cyclic commutativity under the trace. Because convergence is uniform in α we may put the limit before the α-integral. □

In (2.4) the term $P(\alpha)P'(\alpha)P(\alpha)$ was introduced somewhat artificially. We are therefore not surprised that the corresponding term $U(t,\alpha)P(\alpha)BU(t,\alpha)P(\alpha)$ does not contribute to the trace. However, we emphasize that $P(\alpha)P'(\alpha)P(\alpha) = 0$ as an operator whereas here it is only the imaginary part of the trace what vanishes.

Unfortunately, as it stands we cannot characterize the exponent in (4.10) independently of $U(t,\alpha)$ and $P(\alpha)$. An integral equation would require that $U(t,\alpha)$ be integrable on $]-\infty,0]$, which cannot be expected because of the unitarity of $U(t,\alpha)$. Therefore, we again shift the spectrum of $A(\alpha)$ into the complex plane:

$$(4.11) \qquad A_\varepsilon(\alpha) := A(\alpha) + i\varepsilon J,\ J := 2P(0)-\mathbb{1},\ \varepsilon\in\mathbb{R}.$$

Such operators were studied in Sec. 3. We note:

$$(4.12) \qquad J^2 = \mathbb{1},\ J^* = J,\ \|J\| = 1.$$

Since J is bounded, $A_\varepsilon(\alpha)$ is defined on $D(A(\alpha)) = D_0$ and closed since the sum of a closed operator and a bounded operator is again closed. The case $\alpha = 0$ has some special properties.

Lemma 4.5. *Let $A_\varepsilon(0)$ be according to* (4.11). *Then,* $\operatorname{Im}\lambda > 0$ *for* $\lambda\in\sigma_1(A_\varepsilon(0))$ *and* $\operatorname{Im}\lambda < 0$ *for* $\lambda\in\sigma_2(A_\varepsilon(0))$. *Moreover,* $P_\varepsilon(0) = P(0)$.

Proof. Since A_0 and $P(0)$ commute one verifies the identity

$$(4.13)\ (z\mathbb{1} - A_0 - i\varepsilon J)^{-1} = ((z - i\varepsilon)\mathbb{1} - A_0)^{-1} P(0) + ((z + i\varepsilon)\mathbb{1} - A_0)^{-1}(\mathbb{1} - P(0)),$$

which shows immediately the statement about the spectrum. The statement about the spectral projections follows from inserting this identity in (4.3) and taking into account the position of the singularities. □

We want to approximate all quantities in (4.10) by the respective quantities associated with $A_\varepsilon(\alpha)$.

Lemma 4.6. *Let $A_\varepsilon(\alpha)$ be as in (4.11). Then there is an $\varepsilon_0 > 0$ such that $\Gamma \subset \rho(A_\varepsilon(\alpha))$ for all $|\varepsilon| \le \varepsilon_0$ and all $\alpha \in [0,1]$. The spectral projections associated with $\sigma_1(A_\varepsilon(\alpha))$ are given by*

$$P_\varepsilon(\alpha) = \frac{1}{2\pi i} \int_\Gamma R_\varepsilon(z,\alpha)\, dz \tag{4.14}$$

where $R_\varepsilon(z,\alpha) := (z\mathbb{1} - A_\varepsilon(\alpha))^{-1}$. Moreover, $\lim_{\varepsilon\to 0} P_\varepsilon(\alpha) = P(\alpha)$ in the operator norm uniformly for all $\alpha \in [0,1]$.

Proof. Let m be as in (4.2) and pick $\varepsilon_0 > 0$ with $m\varepsilon_0 < 1$. For $z \in \Gamma \subset \rho(A(\alpha))$ a Neumann series argument then shows $R_\varepsilon(z,\alpha)$ exists and is bounded with bound

$$\|R_\varepsilon(z,\alpha)\| \le \frac{m}{1 - \varepsilon_0 m} \tag{4.15}$$

for $|\varepsilon| \le \varepsilon_0$ and all $z \in \Gamma$ and $\alpha \in [0,1]$. Thus, we conclude $\Gamma \subset \rho(A_\varepsilon(\alpha))$. From

$$P_\varepsilon(\alpha) - P(\alpha) = \varepsilon \frac{1}{2\pi} \int_\Gamma R_\varepsilon(z,\alpha) J R(z,\alpha)\, dz$$

we deduce the claimed convergence via (4.15). □

Lemma 4.7. *Let ε be small enough. Then the pseudo inverse $P_\varepsilon^+(\alpha)$ of $P_\varepsilon(\alpha)$ relative to $P(0)$ exists. Moreover, $\lim_{\varepsilon\to 0} P_\varepsilon^+(\alpha) = P^+(\alpha)$ in the operator norm uniformly for $\alpha \in [0,1]$.*

Proof. The statement is only non-trivial on $\operatorname{ran} P(0)$ where a Neumann series argument shows $P_\varepsilon^+(\alpha)$ exists. The uniform convergence follows by an argument as in Lemma 4.6 with m being replaced by $\tilde{m} := \sup_{\alpha\in[0,1]} \|P^+(\alpha)\|$. □

The spectrum of $A_\varepsilon(\alpha)$ can be described in more detail.

Lemma 4.8. *Let $A_\varepsilon(\alpha)$ be as in (4.11). Then for all $\alpha \in [0,1]$*

1. $|\operatorname{Im}\lambda| \le |\varepsilon|$ *for* $\lambda \in \sigma(A_\varepsilon(\alpha))$.
2. *When $\varepsilon \ne 0$ the operator $A_\varepsilon(\alpha)$ is dichotomous.*

3. $\dim \operatorname{ran} P_\varepsilon(\alpha) = N$ *and* $\sigma_1(A_\varepsilon(\alpha))$ *contains exactly* N *eigenvalues counted with multiplicity.*

4. *For* $\varepsilon > 0$ *there is a* $\delta = \delta(\varepsilon) > 0$ *such that* $\operatorname{Im} \lambda \geq \delta$ *for* $\lambda \in \sigma_1(A_\varepsilon(\alpha))$.

Proof. 1. The Neumann series shows, with the aid of the standard estimate of the resolvent and $\|J\| = 1$, that $R_\varepsilon(z, \alpha)$ exists and is bounded when $|\operatorname{Im} z| > |\varepsilon|$. Hence $z \in \rho(A_\varepsilon(\alpha))$.

2. The operator $(\mathbb{1} - P(0))A_0(\mathbb{1} - P(0))$ is self-adjoint on $D_2 = \operatorname{ran}(\mathbb{1} - P(0)) \cap D_0$ and $(\mathbb{1} - P(0))B(\mathbb{1} - P(0))$ is bounded on $\operatorname{ran}(\mathbb{1} - P(0))$ whence we deduce the self-adjointness of $(\mathbb{1} - P(0))A(\alpha)(\mathbb{1} - P(0))$ on D_2. Because of (A4), $A_\varepsilon(\alpha)$ satisfies the dichotomy criterion in Theorem 3.1 and thus has no real spectral values.

3. Lemma 4.6 and (4.5) imply $\dim \operatorname{ran} P_\varepsilon(\alpha) = N$ whence $\sigma_1(A_\varepsilon(\alpha))$ contains exactly N eigenvalues counted with multiplicity.

4. The statement is clear for $\alpha = 0$ by Lemma 4.5 and 3. Since a finite number of eigenvalues depends continuously on α the general statement follows from 2. □

The operator $-iA_\varepsilon(\alpha)$ generates a group of operators $U_\varepsilon(t, \alpha)$, which unlike $U(t, \alpha)$ vanish in some sense for $t \to -\infty$, this being the reason to introduce the parameter ε.

Lemma 4.9. *Let* $A_\varepsilon(\alpha)$ *be as in* (4.11). *Then for all* $\alpha \in [0, 1]$*:*

1. *For all* $\varepsilon \in \mathbb{R}$, $-iA_\varepsilon(\alpha)$ *generates a group* $U_\varepsilon(t, \alpha)$ *with:*

$$\|U_\varepsilon(t, \alpha)\| \leq e^{|\varepsilon t|}, \ t \in \mathbb{R}. \tag{4.16}$$

2. *Let* $\varepsilon > 0$ *and* δ *be as in Lemma 4.8. Then there is* $C = C(\varepsilon) \geq 0$ *such that*

$$\|U_\varepsilon(t, \alpha)P_\varepsilon(\alpha)\| \leq Ce^{\frac{1}{2}\delta t}, \ t \leq 0. \tag{4.17}$$

3. $\lim_{\varepsilon \to 0} U_\varepsilon(t, \alpha) = U(t, \alpha)$ *in the operator norm for each fixed* t *and uniformly in* $\alpha \in [0, 1]$.

Proof. 1. By Lemma 4.8, $|\operatorname{Im} \lambda| \leq |\varepsilon|$ for $\lambda \in \sigma(A_\varepsilon(\alpha))$ whence $-iA_\varepsilon(\alpha)$ generates a group with (4.16) (see [10]).

2. We have by (2.8) after an appropriate change of variables (as in the proof of Lemma 4.3)

$$U_\varepsilon(t, \alpha)P_\varepsilon(\alpha) = \frac{1}{2\pi i} \int_\Gamma e^{-izt} R_\varepsilon(z, \alpha)\, dz. \tag{4.18}$$

From Lemma 4.8, we know that the poles being enclosed by Γ have imaginary part greater or equal δ. We may therefore replace by Cauchy's theorem Γ by Γ_δ with $\operatorname{Im} z \geq \delta/2$ for $z \in \Gamma_\delta$ to derive the desired estimate from (4.18).

3. Semi-group theory (see [10]) provides the estimate

$$\|U_\varepsilon(t, \alpha) - U(t, \alpha)\| \leq e^{|\varepsilon t|} - 1$$

since $U(t, \alpha)$ is unitary. This shows convergence. □

The group $U_\varepsilon(t, 0)$ can be expressed explicitly by $U(t, 0)$.

Lemma 4.10. *Let $U_\varepsilon(t,0)$ be the group generated by $-iA_\varepsilon(0)$. Then,*

$$U_\varepsilon(t,0) = e^{\varepsilon t}U(t,0)P(0) + e^{-\varepsilon t}U(t,0)(\mathbb{1} - P(0)). \tag{4.19}$$

Proof. Insert (4.13) into (2.8) after having changed variables appropriately (as in the proof of Lemma 4.3). □

The groups $U_\varepsilon(t,\alpha)$ and $U_\varepsilon(t,0)$ are related via an integral equation, which will be the source for the Wiener-Hopf equation.

Lemma 4.11. *Let $U_\varepsilon(t,\alpha)$ be the group generated by $-iA_\varepsilon(\alpha)$. Then,*

$$(U_\varepsilon(-t_1,0)U_\varepsilon(t_1,\alpha) - U_\varepsilon(-t_0,0)U_\varepsilon(t_0,\alpha))\varphi = -i\alpha\int_{t_0}^{t_1} U_\varepsilon(-\tau,0)BU_\varepsilon(\tau,\alpha)\varphi\, d\tau. \tag{4.20}$$

Let $\varepsilon > 0$. Then,

$$\begin{aligned}(\mathbb{1} - P(0))&U_\varepsilon(-t_1,0)U_\varepsilon(t_1,\alpha)P_\varepsilon(\alpha)\varphi \\ &= -i\alpha\int_{-\infty}^{t_1}(\mathbb{1} - P(0))U_\varepsilon(-\tau,0)BU_\varepsilon(\tau,\alpha)P_\varepsilon(\alpha)\varphi\, d\tau.\end{aligned} \tag{4.21}$$

Proof. The function $\tau \mapsto U_\varepsilon(-\tau,0)U_\varepsilon(\tau,\alpha)\varphi$ is differentiable for $\varphi \in D_0$ with derivative

$$i\frac{d}{d\tau}U_\varepsilon(-\tau,0)U_\varepsilon(\tau,\alpha)\varphi = \alpha U_\varepsilon(-\tau,0)BU_\varepsilon(\tau,\alpha)\varphi.$$

Integrating from t_0 to t_1 yields (4.20) for $\varphi \in D_0$. Since $U_\varepsilon(t,0)$, $U_\varepsilon(t,\alpha)$, and B are bounded, the formula is valid for all $\varphi \in \mathcal{H}$.

From (4.19) and (4.17), it is seen that $\|(\mathbb{1} - P(0))U_\varepsilon(-t,0)BU_\varepsilon(t,\alpha)P_\varepsilon(\alpha)\| \le \|B\|Ce^{(\frac{\delta}{2}+\varepsilon)t}$ whence we may perform the limit $t_0 \to -\infty$ in (4.20) after having multiplied by $\mathbb{1} - P(0)$ from the left and replaced φ by $P_\varepsilon(\alpha)\varphi$. □

Now we approximate $A(\alpha)$ by $A_\varepsilon(\alpha)$.

Proposition 4.12. *For the spectral projections, there holds the following formula*

$$\begin{aligned}&\det(P(0)P(1)P(0)) \\ &= \lim_{\eta\to+0}\lim_{\varepsilon\to 0}\exp\left[-2\,\mathrm{Re}\, i\int_0^1\int_{-\infty}^0 e^{\eta t}\,\mathrm{tr}\, BU_\varepsilon(t,\alpha)P_\varepsilon(\alpha)P_\varepsilon^+(\alpha)U_\varepsilon(-t,\alpha)\, dt\, d\alpha\right].\end{aligned} \tag{4.22}$$

Proof. We consider the function

$$s_\varepsilon(t) := \sup_{\alpha\in[0,1]} |\,\mathrm{tr}\, BU_\varepsilon(t,\alpha)P_\varepsilon(\alpha)P_\varepsilon^+(\alpha)U_\varepsilon(-t,\alpha) - \mathrm{tr}\, BU(t,\alpha)P(\alpha)P^+(\alpha)U(-t,\alpha)|.$$

Because of the appearance of $P_\varepsilon^+(\alpha)$ and $P^+(\alpha)$, the trace refers only to the finite dimensional subspace $\mathrm{ran}\, P(0)$. Hence we may use Lemmas 4.6, 4.7, and 4.9, which state convergence in the operator norm uniformly in $\alpha \in [0,1]$, to conclude $\lim_{\varepsilon\to 0} s_\varepsilon(t) = 0$ for each fixed $t \leq 0$. From Lemma 4.9 we deduce the estimate

$$|s_\varepsilon(t)| \leq N\|B\| \sup_{\alpha\in[0,1]} \left(e^{2|\varepsilon t|} \|P_\varepsilon(\alpha)\|\|P_\varepsilon^+(\alpha)\| + \|P(\alpha)\|\|P^+(\alpha)\| \right).$$

When $2|\varepsilon| < \eta$, Lebesgue's convergence theorem allows us to interchange the limit with the t-integral. Since we took the supremum over α we may put the limit before the α-integral. □

Note that in the above proposition the limits must not be interchanged. The exponent in (4.22) can now be characterized independently of $U_\varepsilon(t,\alpha)$ and $P_\varepsilon(\alpha)$ through an integral equation. To this end, we introduce the Green operators.

Definition 4.13. The operator defined by $(-\infty < t, t' \leq 0)$

$$G_\varepsilon(t,t';\alpha) := \begin{cases} iU_\varepsilon(t,\alpha)P_\varepsilon(\alpha)P_\varepsilon^+(\alpha)U_\varepsilon(-t',\alpha), & t \leq t', \\ iU_\varepsilon(t,\alpha)[P_\varepsilon(\alpha)P_\varepsilon^+(\alpha) - \mathbb{1}]U_\varepsilon(-t',\alpha), & t > t', \end{cases}$$

is called the time-ordered Green operator.

We note the properties of $G_\varepsilon(t,t';\alpha)$, which are mostly simple reformulations of those of $U_\varepsilon(t,\alpha)$.

Lemma 4.14. *Let $G_\varepsilon(t,t';\alpha)$ be the time-ordered Green operator according to Definition 4.13. Then the function $t \mapsto G_\varepsilon(t,t';\alpha)$ is strongly continuous for all $t \neq t'$ and all $t' \leq 0$. For $-\infty < t \leq t' \leq 0$ and all $\alpha \in [0,1]$ the operators $G_\varepsilon(t,t+0;\alpha)$ and $BG_\varepsilon(t,t+0;\alpha)$ are trace class. Moreover, the limit*

$$\operatorname{tr} BG_\varepsilon(t,t+0;\alpha) := \lim_{\substack{t'\to t\\ t'\geq t}} \operatorname{tr} BG_\varepsilon(t,t';\alpha)$$

exists and the function $\operatorname{tr} BG_\varepsilon(t,t+0;\alpha)$ depends continuously on both t and α.

Let $\varepsilon > 0$. Then,

$$\|G_\varepsilon(t,t';\alpha)\| \leq Ce^{\frac{\delta}{2}t}, \ t \leq t' \leq 0, \tag{4.23}$$

with δ from Lemma 4.8 and a constant $C = C(\varepsilon, t', \alpha) \geq 0$.

Proof. The strong continuity follows immediately from the definition of $G_\varepsilon(t,t';\alpha)$ and the strong continuity of $U_\varepsilon(t,\alpha)$. Since $P_\varepsilon^+(\alpha)$ has finite dimensional range and thus is trace class $G_\varepsilon(t,t+0;\alpha)$ and $BG_\varepsilon(t,t+0;\alpha)$ are trace class, too. The existence and continuity of $\operatorname{tr} BG_\varepsilon(t,t+0;\alpha)$ follow for the same reason. The estimate (4.23) is an immediate consequence of Lemma 4.9. □

The case $\alpha = 0$ has some special features. We define

$$G_0(t) := \begin{cases} iU(t,0)P(0), & t \leq 0, \\ iU(t,0)(P(0) - \mathbb{1}), & t > 0, \end{cases} \tag{4.24}$$

and call $G_0(t)$ free time-ordered Green operator though this is not quite consistent from a notational point of view since one would expect $G_\varepsilon(t,t';0)$ to bear this name. However, it is $U(t,0)$ that is of physical relevance rather than $U_\varepsilon(t,0)$.

Lemma 4.15. *We have* $G_\varepsilon(t,t';0) = e^{-\varepsilon|t-t'|}G_0(t-t')$ *and* $\|G_0(t)\| \le 1$.

Proof. Let $t \le t'$. By Definition 4.13,

$$G_\varepsilon(t,t';0) = iU_\varepsilon(t,0)P_\varepsilon(0)P_\varepsilon^+(0)U_\varepsilon(-t',0),\ t \le t'.$$

From Lemma 4.5, we know $P_\varepsilon(0) = P(0)$ whence we conclude $P_\varepsilon(0)P_\varepsilon^+(0) = P(0)$ by Lemma 2.2. We thus have by Lemma 4.10:

$$G_\varepsilon(t,t';0) = e^{\varepsilon(t-t')}G_0(t-t'),\ t \le t'.$$

For $t > t'$ we obtain analogously

$$G_\varepsilon(t,t';0) = e^{\varepsilon(t'-t)}G_0(t-t').$$

The estimate follows from $U(t,0)$ being unitary and $P(0)$ being an orthogonal projection. □

We check the Wiener-Hopf equation for $G_\varepsilon(t,t';\alpha)$.

Proposition 4.16. *For sufficiently small* $\varepsilon > 0$*, the time-ordered Green operator* $G_\varepsilon(t,t';\alpha)$ *satisfies the Wiener-Hopf equation:*

$$G_\varepsilon(t,t';\alpha)\varphi = G_\varepsilon(t,t';0)\varphi + \alpha\int_{-\infty}^{0} G_\varepsilon(t,\tau;0)BG_\varepsilon(\tau,t';\alpha)\varphi\,d\tau \tag{4.25}$$

with $-\infty < t,t' \le 0$, $\alpha \in [0,1]$, *and* $\varphi \in \mathcal{H}$.

Proof. Let $t \le t'$. Splitting up the integral in (4.25) and inserting the definition of $G_\varepsilon(t,t';\alpha)$ yield

$$\begin{aligned}
I &:= i\alpha\int_{-\infty}^{0} G_\varepsilon(t,\tau;0)BG_\varepsilon(\tau,t';\alpha)\varphi\,d\tau \\
&= -i\alpha\int_{-\infty}^{t} U_\varepsilon(t,0)(P(0)-\mathbb{1})U_\varepsilon(-\tau,0)BU_\varepsilon(\tau,\alpha)P_\varepsilon(\alpha)P_\varepsilon^+(\alpha)U_\varepsilon(-t',\alpha)\varphi\,d\tau \\
&\quad - i\alpha\int_{t}^{t'} U_\varepsilon(t,0)P(0)U_\varepsilon(-\tau,0)BU_\varepsilon(\tau,\alpha)P_\varepsilon(\alpha)P_\varepsilon^+(\alpha)U_\varepsilon(-t',\alpha)\varphi\,d\tau \\
&\quad - i\alpha\int_{t'}^{0} U_\varepsilon(t,0)P(0)U_\varepsilon(-\tau,0)BU_\varepsilon(\tau,\alpha)(P_\varepsilon(\alpha)P_\varepsilon^+(\alpha)-\mathbb{1})U_\varepsilon(-t',\alpha)\varphi\,d\tau.
\end{aligned}$$

We use the integral equation from Lemma 4.11:

$$\begin{aligned}
I &= U_\varepsilon(t,0)(P(0)-\mathbb{1})U_\varepsilon(-t,0)U_\varepsilon(t,\alpha)P_\varepsilon(\alpha)P_\varepsilon^+(\alpha)U_\varepsilon(-t',\alpha)\varphi \\
&\quad + U_\varepsilon(t,0)P(0)[U_\varepsilon(-t',0)U_\varepsilon(t',\alpha) - U_\varepsilon(-t,0)U_\varepsilon(t,\alpha)]P_\varepsilon(\alpha)P_\varepsilon^+(\alpha)U_\varepsilon(-t',\alpha)\varphi \\
&\quad + U_\varepsilon(t,0)P(0)[\mathbb{1} - U_\varepsilon(-t',0)U_\varepsilon(t',\alpha)](P_\varepsilon(\alpha)P_\varepsilon^+(\alpha)-\mathbb{1})U_\varepsilon(-t',\alpha)\varphi \\
&= -U_\varepsilon(t,0)U_\varepsilon(-t,0)U_\varepsilon(t,\alpha)P_\varepsilon(\alpha)P_\varepsilon^+(\alpha)U_\varepsilon(-t',\alpha)\varphi \\
&\quad + U_\varepsilon(t,0)P(0)(P_\varepsilon(\alpha)P_\varepsilon^+(\alpha)-\mathbb{1})U_\varepsilon(-t',\alpha)\varphi \\
&\quad + U_\varepsilon(t,0)P(0)U_\varepsilon(-t',0)U_\varepsilon(t',\alpha)U_\varepsilon(-t',\alpha)\varphi
\end{aligned}$$

A look at Lemma 2.2 shows that the middle term vanishes:

$$\begin{aligned} I &= -U_\varepsilon(t,\alpha)P_\varepsilon(\alpha)P_\varepsilon^+(\alpha)\varphi U_\varepsilon(-t',\alpha)\varphi + U_\varepsilon(t,0)P(0)U_\varepsilon(-t',0)\varphi \\ &= iG_\varepsilon(t,t';\alpha)\varphi - iG_\varepsilon(t,t';0)\varphi. \end{aligned}$$

This yields (4.25) for $t \le t'$. For $t > t'$ we again split up the integral according to $-\infty < t' < t \le 0$. The proof is then completed by analogous calculations. □

We obtain this paper's main result by simply collecting what has been proved so far.

Theorem 4.17. *(Adiabatic theorem) Let $A_0 : D(A_0) \to \mathcal{H}$ be a self-adjoint and $B : \mathcal{H} \to \mathcal{H}$ a bounded self-adjoint operator such that the operator pencil $A(\alpha) := A_0 + \alpha B$ satisfies conditions (A1) through (A4). Denote by $G_0(t)$ the free time-ordered Green operator according to* (4.24)*. Finally, let $\varepsilon > 0$. Then the Wiener-Hopf equation*

$$G_\varepsilon(t,t';\alpha)\varphi = e^{-\varepsilon|t-t'|}G_0(t-t')\varphi + \alpha\int_{-\infty}^{0} e^{-\varepsilon|t-\tau|}G_0(t-\tau)BG_\varepsilon(\tau,t';\alpha)\varphi\,d\tau \tag{4.26}$$

with $\varphi \in \mathcal{H}$, $-\infty < t,t' \le 0$, and $\alpha \in [0,1]$ possesses at least one solution $G_\varepsilon(t,t';\alpha)$, the time-ordered Green operator, having the properties:

1. *For $-\infty < t,t \le 0$, and all $\alpha \in [0,1]$ the function $t \mapsto G_\varepsilon(t,t';\alpha)$ is strongly continuous when $t \ne t'$ and satisfies the estimate*

   $$\|G_\varepsilon(t,t';\alpha)\| \le Ce^{\frac{\delta}{2}},\ \ t \le t' \le 0, \tag{4.27}$$

   with constants $C = C(\varepsilon,t',\alpha) \ge 0$ and $\delta = \delta(\varepsilon) > 0$.

2. *For $t \le t' \le 0$ the operators $G_\varepsilon(t,t';\alpha)$ and $BG_\varepsilon(t,t';\alpha)$ are trace class and the function*

   $$\operatorname{tr} BG_\varepsilon(t,t+0;\alpha) := \lim_{\substack{t'\to t\\ t'\ge t}} \operatorname{tr} BG_\varepsilon(t,t';\alpha) \tag{4.28}$$

   is well defined and continuous with respect to both t and α.

3. *The section determinants can be expressed by the formula:*
   (4.29)
   $$\det(P(0)P(1)P(0)) = \lim_{\eta\to+0}\lim_{\varepsilon\to+0}\exp\left[-\int_0^1\int_{-\infty}^0 e^{\eta t}\operatorname{tr} BG_\varepsilon(t,t+0;\alpha)\,dt\,d\alpha\right].$$

Proof. Combine Propositions 4.12 and 4.16 and use Lemma 4.15 in order to reduce $G_\varepsilon(t,t';0)$. □

5. Solution theory of the Wiener-Hopf equation

We examine the Wiener-Hopf equation (4.22). We shall show that the homogeneous equation is connected with a homogeneous boundary value problem. The solution theory is in some sense analogous to a similar equation discussed in [8]. In this section, the parameter α is not crucial. We retain it to stay in concordance with the notation of Sec. 4.

The time-ordered Green operator $G_\varepsilon(t,t';\alpha)$ satisfies, at least formally, a differential equation:

$$i\frac{\partial}{\partial t}G_\varepsilon(t,t';\alpha)=\delta(t-t')\mathbb{1}+A_\varepsilon(\alpha)G_\varepsilon(t,t';\alpha). \tag{5.1}$$

Here δ is the Dirac delta. Furthermore, we have simple boundary conditions.

Lemma 5.1. *The time-ordered Green operator according to Definition* 4.13 *satisfies for all* $\alpha\in[0,1]$ *the boundary conditions*

$$P(0)G_\varepsilon(0,t';\alpha)=0,\ \lim_{t\to-\infty}G_\varepsilon(t,t';\alpha)\varphi=0,\ -\infty<t'\le 0. \tag{5.2}$$

Proof. Recall $P(0)P_\varepsilon(\alpha)P_\varepsilon^+(\alpha)=P(0)$ from Lemma 2.2 to conclude:

$$P(0)G_\varepsilon(0,t';\alpha)=iP(0)U_\varepsilon(0,\alpha)[P_\varepsilon(\alpha)P_\varepsilon^+(\alpha)-\mathbb{1}]U_\varepsilon(-t',\alpha)=0.$$

The second condition is clear from Lemma 4.14. □

Equation (5.1) and Lemma 5.1 justify the name Green operator and motivate the following theorem.

Theorem 5.2. *Let the function* $\varphi(\cdot):t\mapsto\varphi(t)\in\mathcal{H}$ *be continuously differentiable and solve the integral equation:*

$$\varphi(t)=\alpha\int_{-\infty}^{0}G_\varepsilon(t,\tau;0)B\varphi(\tau)\,d\tau,\ -\infty<t\le 0, \tag{5.3}$$

with $\int_{-\infty}^{0}\|\varphi(\tau)\|\,d\tau<\infty$ *and* $\lim_{t\to-\infty}\|\varphi(t)\|=0$. *Then* $\varphi(\cdot)$ *also solves the boundary value problem:*

$$i\frac{d}{dt}\varphi(t)=A_\varepsilon(\alpha)\varphi(t),\ P(0)\varphi(0)=0,\ \varphi(-\infty)=0. \tag{5.4}$$

Proof. We split up the integral in (5.3):

$$\begin{aligned}I(t)&:=\int_{-\infty}^{0}G_\varepsilon(t,\tau;0)B\varphi(\tau)\,d\tau\\&=\int_{-\infty}^{t}iU_\varepsilon(t-\tau,0)(P(0)-\mathbb{1})B\varphi(\tau)\,d\tau+\int_{t}^{0}iU_\varepsilon(t-\tau,0)P(0)B\varphi(\tau)\,d\tau.\end{aligned}$$

We have assumed $\varphi(\cdot)$ to be continuously differentiable. Then it is known from semigroup theory that the respective integrands are also continuously differentiable since all operators involved are bounded. Moreover, $I(\cdot)$ is continuously differentiable and $\varphi(\cdot)$ solves the differential equation (5.4). The boundary conditions are clear from Lemma 5.1. □

The assumption that the solution to the integral equation is continuously differentiable generally does not follow from the integral equation when the operator A_0 is unbounded (see [10], Sec. 4.2) and therefore cannot be dropped. The converse of Theorem 5.2 can be shown.

Theorem 5.3. *Let the function $\varphi(\cdot) : t \mapsto \varphi(t) \in \mathcal{H}$ be continuously differentiable. Moreover, let $\varphi(0) \in D(A_\varepsilon(\alpha))$ and $\int_{-\infty}^{0} \|\varphi(\tau)\| \, d\tau < \infty$. If $\varphi(\cdot)$ is a solution to the boundary value problem* (5.4) *then $\varphi(\cdot)$ satisfies the integral equation* (5.3)*.*

Proof. Let $\varphi(\cdot)$ be a solution to the boundary value problem (5.4). Then,

$$\varphi(t) = U_\varepsilon(t, \alpha)\varphi(0)$$

(see [10]). Multiply (4.20) in Lemma 4.11 by $P(0)$ and $\mathbb{1} - P(0)$, respectively, to obtain:

$$P(0)\varphi(t) = 0 + i\alpha \int_t^0 P(0)U_\varepsilon(t-\tau, 0)B\varphi(\tau)\, d\tau \tag{5.5}$$

$$(\mathbb{1} - P(0))\varphi(t) = i\alpha \int_{-\infty}^t (P(0) - \mathbb{1})U_\varepsilon(t-\tau, 0)B\varphi(\tau)\, d\tau \tag{5.6}$$

where we used the boundary condition (5.4). Adding (5.5) and (5.6) yields the statement. □

At this point the analogy to the solution theory developed in [8] breaks down because we cannot relate without further effort the solvability of the boundary value problem with the determinant $\det(P(0)P_\varepsilon(\alpha)P(0))$. This is due to the fact that our problem is not, strictly speaking, a boundary value problem because the second boundary point is missing, i.e. the solution vanishing at infinity is not really a boundary condition.

6. A uniqueness criterion

In the preceding section, we showed the equivalence of the integral equation (5.3) and the boundary value problem (5.4). However, we have not yet given a practical criterion that ensures that either of the above problems only has the trivial solution. This will be done now for the boundary value problem.

For the sake of the reader's convenience, we recall the boundary value problem under consideration. Let $P : \mathcal{H} \to \mathcal{H}$ be an orthogonal projection, $J := 2P - \mathbb{1}$, and $A : \mathcal{H} \to \mathcal{H}$ be a bounded, self-adjoint operator. The boundedness of A is not really essential. We require it in order to heighten the clarity of the presentation. Now,

$$i\frac{d}{dt}\varphi(t) = (A + i\varepsilon J)\varphi(t), \; P\varphi(0) = 0, \; \varphi(-\infty) = 0, \tag{6.1}$$

where the function $\varphi(\cdot) : t \mapsto \varphi(t) \in \mathcal{H}$ is continuously differentiable and absolutely integrable on $]-\infty, 0]$ and $\varepsilon \in \mathbb{R}$. We write $\mathcal{H}$ as orthogonal sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with $\mathcal{H}_1 := \operatorname{ran} P$ and $\mathcal{H}_2 := \mathcal{H}_1^\perp$ and represent A and J as block matrices:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \; A_{jk} : \mathcal{H}_k \to \mathcal{H}_j, \; j,k = 1,2, \; J = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{6.2}$$

Note that the self-adjointness of A implies $A_{jk}^* = A_{kj}$, $j,k = 1,2$. If we had $A_{12} = 0$ and $A_{21} = 0$ then $\mathcal{H}_1$ would be an invariant subspace for A and the boundary value problem (6.1) would easily be seen to have only the trivial solution. In general, this will not be the case. However, instead of being furnished with an invariant subspace, we just need that A satisfies the conditions in Theorem 3.1.

Theorem 6.1. *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded, self-adjoint operator and $\gamma > 0$. Assume either*

$$A_{11} \leq 0,\ \gamma\mathbb{1} \leq A_{22} \text{ or } A_{11} \geq 0,\ -\gamma\mathbb{1} \geq A_{22} \tag{6.3}$$

in the sense of quadratic forms. When $\varepsilon \geq 0$, the boundary value problem (6.1) *possesses only the trivial solution.*

Proof. Let $\varphi(t)$ be a solution to the boundary value problem (6.1). We write $\varphi = (\varphi_1, \varphi_2)^T$ according to $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. By noting the self-adjointness of A and J we see

$$i\frac{d}{dt}(\varphi(t), A\varphi(t)) = \varepsilon(\varphi(t), (AJ + JA)\varphi(t)).$$

Integrating yields:

$$(\varphi_2(0), A_{22}\varphi_2(0)) = \varepsilon \int_{-\infty}^{0} (\varphi(t), (AJ + JA)\varphi(t))\, dt$$

where we used the boundary conditions. Now,

$$AJ + JA = 2\begin{pmatrix} A_{11} & 0 \\ 0 & -A_{22} \end{pmatrix}$$

whence we obtain:

$$(\varphi_2(0), A_{22}\varphi_2(0)) + 2\varepsilon \int_{-\infty}^{0} (\varphi_2(t), A_{22}\varphi_2(t))\, dt = 2\varepsilon \int_{-\infty}^{0} (\varphi_1(t), A_{11}\varphi_1(t))\, dt.$$

By assumption, the left-hand side is non-negative (non-positive) and the right-hand side is non-positive (non-negative), which implies $\varphi_2(0) = 0$. This shows $\varphi(t) = 0$ since $\varphi(t) = U_\varepsilon(t)\varphi(0)$ with the group $U_\varepsilon(t)$ generated by $-i(A + i\varepsilon J)$ (see [10]). □

It is easy to extend Theorem 6.1 to more general gap conditions.

Corollary 6.2. *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded, self-adjoint operator and $\beta < \gamma$. Assume either*

$$A_{11} \leq \beta\mathbb{1},\ \gamma\mathbb{1} \leq A_{22} \text{ or } A_{11} \geq -\beta\mathbb{1},\ -\gamma\mathbb{1} \geq A_{22}, \tag{6.4}$$

in the sense of quadratic forms. When $\varepsilon \geq 0$, the boundary value problem (6.1) *possesses only the trivial solution.*

Proof. Apply Theorem 6.1 to the operators $A - \beta\mathbb{1}$ and $A + \beta\mathbb{1}$, respectively. □

References

[1] ANDERSON, P.W., Infrared catastrophe in Fermi gases with local scattering potentials, Phys. Rev. Lett. **18**, No. 24, 1049-1051, 1967

[2] ATKINSON, F.V., LANGER, H., MENNICKEN, R., and SHKALIKOV, A.A., The essential spectrum of some matrix operators, Math. Nachr. **167**, 5-20, 1994

[3] AVRON, J.E., SEILER, R., and YAFFE, L.G., Adiabatic Theorems and Applications to the Quantum Hall Effect, Comm. Math. Phys. **110**, 33-49, 1987

[4] BART, H., GOHBERG, I., and KAASHOEK, M.A., Fredholm theory of Wiener-Hopf equations in terms of realization of their symbols, Integral Equations Operator Theory **8**, 590-613, 1985

[5] BASKAKOV, A.G., Dichotomy of the spectrum of non-self-adjoint operators, Siberian Math. J. **32**, No.3, 370-375, 1991; translation from Sibirsk. Mat. Zh. **32**, No. 3 (187), 24-30, 1991

[6] KATO, T., Perturbation theory for linear operators, Springer Verlag, Berlin, 1966

[7] LANGER, H. and TRETTER, C., Diagonalization of certain block operator matrices and applications to Dirac operators, Operator theory and analysis (Amsterdam, 1997), 331-358, Oper. Theory Adv. Appl., Vol. 122, Birkäuser, Basel, 2001

[8] OTTE, P., An integral formula for section-determinants of semi-groups of linear operators, J. Phys. A **32**, 3793-3803, 1999

[9] OTTE, P., Integralformeln für Abschnittsdeterminanten linearer Operatoren (Integral formulas for section determinants of linear operators), PhD thesis, University of Göttingen, 1998

[10] PAZY, A., Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer Verlag, New York, 1983

[11] PRÖSSDORF, S., Some classes of singular equations, North-Holland Mathematical Library, 17, North-Holland Publishing Co., Amsterdam-New York, 1978

[12] REED, M. and SIMON, B., Methods of Modern Mathematical Physics I: Functional Analysis, Academic Press, New York, 1972

[13] RIVIER, N. and SIMANEK, E., Exact Calculation of the Orthogonality Catastrophe in Metals, Phys. Rev. Lett. **26**, No. 8, 435-438, 1971

Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstraße 150
D-44780 Bochum
Germany
E – mail:
peter.otte@ruhr-uni-bochum.de