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Integral Equations and Operator Theory

A Fredholm Determinant Formula for Section Determinants of Bounded Operators

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Abstract. We show that the section determinant of e^A can be expressed, under certain conditions, by the Fredholm determinant of an integral operator. The kernel function of this integral operator is computed explicitly in terms of the operator A. As a simple consequence we derive a Weierstrass type product expansion for the section determinant.

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1. Introduction

Section determinants of linear operators occur, e.g., in quantum mechanics where they describe transition probabilities within fermionic many-particle systems. Our aim here is to relate the investigation of those determinants to several other problems such as eigenvalue problems of integral operators and boundary value problems of evolution equations (see Lemmas 3.3 through 3.5). This is done mainly by showing that the section determinant can be written essentially as Fredholm determinant of some integral operator.

To be more precise let $A : \mathcal{H} \to \mathcal{H}$ be a bounded operator acting on the Hilbert space $\mathcal{H}$ and $U := e^A$ be the exponential function of A. Let $\mathcal{H}$ be written as orthogonal sum $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$ and represent U by a 2×2 block matrix corresponding to this decomposition

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}.$$

Assume that $\dim \mathcal{H}_1 < \infty$ and $\mathcal{H}_2$ is separable. The section determinant $\det U_{11}$ then is well-defined. Under some conditions to be imposed on A we show in Corollary 3.7 that there holds the equality

$$\det U_{11} = e^{\operatorname{tr} A_{11}} \det(\mathbb{1} + \hat{K}_0) \tag{1}$$

where we have decomposed A analogously to U. The integral operator $\hat{K}_0$ acts on $L^2([0,1], \mathcal{H}_1)$. It is trace class and its kernel function $K_0(t,t')$ is given by

$$K_0(t,t') = \int_{\max\{t,t'\}}^1 e^{(t-\tau)A_{11}} A_{12} e^{(\tau-t')A_{22}} A_{21}\, d\tau.$$

This easily implies that if $U(T) := e^{TA}$, then

$$\det U_{11}(T) = e^{T \operatorname{tr} A_{11}} \det(\mathbb{1} + \hat{K}_{0,T}), \tag{2}$$

where $\hat{K}_{0,T}$ is the integral operator on the space $L^2([0,T], \mathcal{H}_1)$ with the kernel function

$$K_{0,T}(t,t') = \int_{\max\{t,t'\}}^T e^{(t-\tau)A_{11}} A_{12} e^{(\tau-t')A_{22}} A_{21}\, d\tau.$$

For instance, formula (1) is valid when A is self-adjoint as is shown in Section 4. The main ingredient to this result is an integral formula due to the author [4], which expresses section determinants through the solution to some integral equation. Integral formulae of this type originally appeared in the physics literature in the framework of many-particle physics where section determinants describe transition probabilities in fermionic systems. However, the derivations given therein are highly formal and cannot be made rigorous with a reasonable expenditure.

A related question was discussed in the framework of section determinants of Toeplitz matrices. In 1999 A. Its and P. Deift asked whether such determinants could be interpreted as Fredholm determinants of operators having some special properties. This was answered affirmatively by Borodin and Okounkov [1] (see also [2]). However, their formula intensively exploits the Toeplitz structure of the operators considered which makes it inapplicable to our more general issue.

2. An integral formula

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be complex Hilbert spaces with $N := \dim \mathcal{H}_1 < \infty$ and $\mathcal{H}_2$ being separable. We consider the orthogonal sum

$$\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2. \tag{3}$$

Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. The boundedness of A is not really necessary but it helps to avoid annoying technicalities that would only detract from the main ideas. A can be written as block matrix corresponding to the decomposition (3)

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{4}$$

with $A_{jk} : \mathcal{H}_k \to \mathcal{H}_j$, $j, k = 1, 2$. All block matrices that will appear subsequently are to be understood with respect to (3). Let $U(t) : \mathcal{H} \to \mathcal{H}$, $t \in \mathbb{R}$, be given by $U(t) := e^{tA}$, where the exponential is defined via the usual power series. Note $U(t_1+t_2) = U(t_1)U(t_2)$. In the context of semi-group theory $U(t)$ is also called the group of bounded linear operators generated by A. Since A is bounded $U(t)$ depends continuously differentiably on t with respect to the operator norm. Here and in what follows analysis of operator-valued functions is always meant with respect to the operator norm. We are interested in the section determinant $\det U_{11}(1)$, which is well-defined because of $N < \infty$. We want to use the integral formula in [4]. To this end we decompose A into

$$A = A_0 + B,\ A_0 := \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},\ B := \begin{pmatrix} 0 & 0 \\ A_{21} & 0 \end{pmatrix}. \tag{5}$$

We first compute $U_0(t) := e^{tA_0}$ because this is considered the unperturbed problem and thus ought to be explicitly soluble. We determine $U_0(t)$ via the initial value problem

$$U_0' = A_0 U_0,\ U_0(0) = \mathbb{1} \tag{6}$$

or written in components

$$\begin{array}{rcccc} U_{0,1j}' & = & A_{11}U_{0,1j} & + & A_{12}U_{0,2j} \\ U_{0,2j}' & = & & & A_{22}U_{0,2j} \end{array} \tag{7}$$

with $j = 1, 2$. It is easily checked that the differential equations are solved by

$$\begin{aligned} U_{0,2j}(t) &= e^{tA_{22}}U_{0,2j}(0) \\ U_{0,1j}(t) &= e^{tA_{11}}U_{0,1j}(0) + \int_0^t e^{(t-\tau)A_{11}}A_{12}e^{\tau A_{22}}U_{0,2j}(0)\,d\tau. \end{aligned} \tag{8}$$

Now the initial conditions yield

$$U_0(t) = \begin{pmatrix} e^{tA_{11}} & \int_0^t e^{(t-\tau)A_{11}}A_{12}e^{\tau A_{22}}\,d\tau \\ 0 & e^{tA_{22}} \end{pmatrix}. \tag{9}$$

Finally, we introduce the free time-ordered Green operator $G_0(t, t')$ by

$$G_0(t,t') := \begin{cases} U_0(t)U_0^+(1)U_0(1-t') & t \le t', \\ U_0(t)U_0^+(1)U_0(1-t') - U_0(t-t') & t > t' \end{cases} \tag{10}$$

where

$$U_0^+(1) := \begin{pmatrix} U_{0,11}^{-1}(1) & 0 \\ 0 & 0 \end{pmatrix} \tag{11}$$

is the so-called pseudo-inverse of $U_0(1)$ with respect to $\mathcal{H}_1$. Now we have collected the necessary prerequisites to write down the announced integral formula.

Proposition 2.1. *Let $A(\alpha) := A_0 + \alpha B$, $\alpha \in [0,1]$, and $U(1,\alpha) := e^{A(\alpha)}$. We assume $\det U_{11}(1,\alpha) \neq 0$ for all $\alpha \in [0,1]$. Then the integral equation*

$$G(t,t';\alpha) = G_0(t,t') - \alpha\int_0^1 G_0(t,\tau)BG(\tau,t';\alpha)\,d\tau,\ \ t,t',\alpha \in [0,1], \tag{12}$$

possesses exactly one solution $G(t,t';\alpha)$, the time-ordered Green operator, having the following properties.

1. *For each $t' \in [0,1]$ and each $\alpha \in [0,1]$, the function $t \mapsto G(t,t';\alpha)$ is continuous on $[0,1]$ for $t \neq t'$.*
2. *For $0 \leq t \leq t' \leq 1$ the operators $G(t,t';\alpha)$ and $BG(t,t';\alpha)$ are trace class and the function*

$$\operatorname{tr} BG(t,t+0;\alpha) := \lim_{\substack{t' \to t \\ t' \geq t}} \operatorname{tr} B(G(t,t';\alpha)$$

   is well-defined and continuous with respect to both t and α.

Moreover, the section determinant can be expressed by

$$\det U_{11}(1) = e^{\operatorname{tr} A_{11}} \exp\left[\int_0^1 \int_0^1 \operatorname{tr} BG(t,t+0;\alpha)\,dt\,d\alpha\right]. \tag{13}$$

Proof. [4], Theorem 2.9 and Corollary 3.5. □

The expression $\int_0^1 \operatorname{tr} BG(t,t+0;\alpha)\,dt$ looks like the trace of the integral operator defined by

$$\varphi(t) \mapsto \int_0^1 BG(t,\tau;\alpha)\varphi(\tau)\,d\tau.$$

Note that this trace is not to be confounded with that referring to $\mathcal{H}$. Unfortunately, the above integral operator is not trace class because of the singularity of the kernel, which is easily deduced from the integral equation (12) or the explicit expression given in [4]. However, a closer look at (13) shows

$$\operatorname{tr} BG = \operatorname{tr} A_{21}G_{12} \tag{14}$$

whence we only need to know G_{12}. The integral equation (12) for G_{12} reads

$$G_{12}(t,t';\alpha) = G_{0,12}(t,t') - \alpha \int_0^1 G_{0,12}(t,\tau)A_{21}G_{12}(\tau,t';\alpha)\,d\tau. \tag{15}$$

Now it is time to compute the explicit form of $G_{0,12}(t,t')$. Let $t \leq t'$. Then,

$$\begin{aligned} G_{0,12}(t,t') &= U_{0,11}(t)U_{0,11}^{-1}(1)U_{0,12}(1-t') \\ &= e^{(t-1)A_{11}}e^{(1-t')A_{11}} \int_0^{1-t'} e^{-\tau A_{11}}A_{12}e^{\tau A_{22}}\,d\tau \\ &= e^{(t-t')A_{11}} \int_{t'}^1 e^{-(\tau-t')A_{11}}A_{12}e^{(\tau-t')A_{22}}\,d\tau \\ &= \int_{t'}^1 e^{(t-\tau)A_{11}}A_{12}e^{(\tau-t')A_{22}}\,d\tau. \end{aligned}$$

For $t > t'$ the only difference comes from $U_{0,12}(t-t')$ (see (10))

$$\begin{aligned} U_{0,12}(t-t') &= e^{(t-t')A_{11}} \int_0^{t-t'} e^{-\tau A_{11}} A_{12} e^{\tau A_{22}}\, d\tau \\ &= e^{(t-t')A_{11}} \int_{t'}^{t} e^{-(\tau-t')A_{11}} A_{12} e^{(\tau-t')A_{22}}\, d\tau. \end{aligned}$$

Hence,

$$G_{0,12}(t,t') = \int_t^1 e^{(t-\tau)A_{11}} A_{12} e^{(\tau-t')A_{22}}\, d\tau.$$

In a more compact form $G_{0,12}(t,t')$ reads

$$G_{0,12}(t,t') = \int_{\max\{t,t'\}}^1 e^{(t-\tau)A_{11}} A_{12} e^{(\tau-t')A_{22}}\, d\tau. \tag{16}$$

We can now rewrite formula (13).

Proposition 2.2. *Let* $\det U_{11}(1,\alpha) \neq 0$ *for* $\alpha \in [0,1]$ *and define*

$$K_0(t,t') = \int_{\max\{t,t'\}}^1 e^{(t-\tau)A_{11}} A_{12} e^{(\tau-t')A_{22}} A_{21}\, d\tau. \tag{17}$$

Then the integral equation

$$K(t,t';\alpha) = K_0(t,t') - \alpha \int_0^1 K_0(t,\tau) K(\tau,t';\alpha)\, d\tau,\ \ t,t',\alpha \in [0,1], \tag{18}$$

admits a solution $K(t,t';\alpha)$ *having the following properties.*

1. *For each* $t' \in [0,1]$ *and each* $\alpha \in [0,1]$*, the function* $t \mapsto K(t,t';\alpha)$ *is continuous on* $[0,1]$.
2. *The operator* $K(t,t;\alpha) : \mathcal{H}_1 \to \mathcal{H}_1$ *is trace class and depends continuously on both* t *and* α.

Furthermore, with the aid of this solution the section determinant can be expressed by the formula

$$\det U_{11}(1) = e^{\operatorname{tr} A_{11}} \exp\left[\int_0^1 \int_0^1 \operatorname{tr} K(t,t;\alpha)\, dt\, d\alpha\right]. \tag{19}$$

Proof. Note that $\operatorname{tr} A_{21}G_{12} = \operatorname{tr} G_{12}A_{21}$ in (14). We know that (12) is uniquely solvable. In particular G_{12} is uniquely determined. Multiplying (15) from the right by A_{21} and putting

$$K_0(t,t') := G_{0,12}(t,t')A_{21},\ \ K(t,t';\alpha) := G_{12}(t,t';\alpha)A_{21}$$

yield the integral equation (18). The explicit expression for K_0 follows from (16). The continuity of $K(\cdot,t';\alpha)$ follows from that of $K_0(\cdot,t')$. Since $\mathcal{H}_1$ is finite dimensional $K(t,t';\alpha)$ being trace class is implied by the boundedness of $G_{0,12}(t,t') : \mathcal{H}_2 \to \mathcal{H}_1$. □

Without uniqueness of the solution to (18) the above proposition is not very helpful because we are not told how to choose the correct solution to be inserted in (19). One way to show uniqueness is to prove that (18) is uniquely solvable if and only if (12) is and then use Proposition 2.1. Although this is not too difficult we prefer, however, to study independently the solvability of (18) in the next section because we consider this to be more instructive.

3. The operator version of the integral formula

Since we want to find an integral operator $\hat{K}(\alpha)$ such that the trace of $\hat{K}(\alpha)$ equals the inner integral in (19) we first need an appropriate Hilbert space on which it can act. We take $\hat{\mathcal{H}} := L^2([0,1], \mathcal{H}_1)$, i.e. the space of square integrable functions having values in $\mathcal{H}_1$. Analogously to the operator-valued case, analysis of vector-valued functions always refers to the corresponding Hilbert space norm. Now let $\hat{K}(\alpha)$ and $\hat{K}_0$ be defined as follows:

$$(\hat{K}(\alpha)\hat{\varphi})(t) := \int_0^1 K(t,\tau;\alpha)\hat{\varphi}(\tau)\,d\tau,\ (\hat{K}_0\hat{\varphi})(t) := \int_0^1 K_0(t,\tau)\hat{\varphi}(\tau)\,d\tau \qquad (20)$$

with $\hat{\varphi} \in \hat{\mathcal{H}}$. It is easy to see that $\hat{K}_0$ and $\hat{K}(\alpha)$ are well-defined bounded linear operators mapping $\hat{\mathcal{H}}$ into itself. Since the kernel functions are continuous we may hope that we are faced with trace class operators. In order to verify this we need some well-known facts from operator theory, which may be found in [3].

Lemma 3.1. *Let $\mathcal{K}$ be a separable Hilbert space.*

1. *Let $L, M : \mathcal{K} \to \mathcal{K}$ be Hilbert-Schmidt operators. Then $LM : \mathcal{K} \to \mathcal{K}$ is a trace class operator.*
2. *Let $L(t,t')$ be a square integrable function from some separable L^2-space, i.e.*
$$\int_0^1 \int_0^1 \|L(t,t')\|^2\,dt\,dt' < \infty.$$
Then the corresponding integral operator having $L(t,t')$ as its kernel function is Hilbert-Schmidt.
3. *Let $L : \mathcal{K} \to \mathcal{K}$ be trace class and $M : \mathcal{K} \to \mathcal{K}$ be bounded. Then LM and ML are trace class, too.*

Proof. See [3]. Chap. III, Remark 7.1; Chap. III, Sec. 9, P. 109; Chap. III, Theorem 7.1. □

Equipped with the above lemma we can prove that at least the operator $\hat{K}_0$ fulfills our hopes.

Proposition 3.2. *The operator $\hat{K}_0 : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ is trace class.*

Proof. Define $\hat{L}$ and $\hat{M}$ via their kernel functions

$$L(t,t') = \Theta(t'-t)e^{(t-t')A_{11}} \text{ and } M(t,t') = \Theta(t-t')A_{12}e^{(t-t')A_{22}}A_{21},$$

respectively. Here Θ is the Heaviside function. It is a straightforward task to check that $\hat{K}_0 = \hat{L}\hat{M}$. It is also clear that

$$\int_0^1 \int_0^1 \|L(t,t')\|^2 \, dt \, dt' < \infty \text{ and } \int_0^1 \int_0^1 \|M(t,t')\|^2 \, dt \, dt' < \infty$$

whence we conclude with the aid of Lemma 3.1 that $\hat{L}$ and $\hat{M}$ are Hilbert-Schmidt. The same lemma then yields that $\hat{K}_0$ is trace class. $\square$

We want $\hat{K}(\alpha)$ to be trace class, too. In order to find conditions under which this is true we notice that the integral equation (18) can be read as operator equation for $\hat{K}(\alpha)$ and $\hat{K}_0$

$$\hat{K}(\alpha) = \hat{K}_0 - \alpha \hat{K}_0 \hat{K}(\alpha). \tag{21}$$

If $(\mathbb{1} + \alpha\hat{K}_0)^{-1}$ exists and is bounded we can solve for $\hat{K}(\alpha) = (\mathbb{1} + \alpha\hat{K}_0)^{-1}\hat{K}_0$ and deduce from Lemma 3.1 that $\hat{K}(\alpha)$ is trace class. Therefore, we examine the spectrum $\sigma(\hat{K}_0)$ of $\hat{K}_0$ which is not only useful for our present needs but may also become important for estimating the trace of $\hat{K}(\alpha)$.

Since $\hat{K}_0$ is a compact operator on an infinite dimensional space we know $0 \in \sigma(\hat{K}_0)$. We investigate the non-zero eigenvalues by showing that the eigenvalue equation is equivalent to a boundary value problem for a differential equation.

Lemma 3.3. *For any $\lambda \neq 0$ the eigenvalue equation*

$$\int_0^1 K_0(t,\tau)\hat{\varphi}(\tau)\, d\tau = \lambda \hat{\varphi}(t) \tag{22}$$

has a non-trivial solution $\hat{\varphi} \in \hat{\mathcal{H}}$ if and only if there is a non-trivial solution to the boundary value problem

$$\begin{pmatrix} \varphi_1'(t) \\ \varphi_2'(t) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ -\frac{1}{\lambda}A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix} \text{ and } \varphi_1(1) = 0,\ \varphi_2(0) = 0. \tag{23}$$

with $\varphi_j(\cdot) : t \mapsto \varphi_j(t) \in \mathcal{H}_j$, $j = 1, 2$, being continuously differentiable.

Proof. Let $\hat{\varphi}$ be an eigenfunction of $\hat{K}_0$ to a non-zero eigenvalue λ. We define

$$\varphi_1(t) := \hat{\varphi}(t),\ \varphi_2(t) := -\frac{1}{\lambda}\int_0^t e^{(t-\tau)A_{22}} A_{21}\varphi_1(\tau)\, d\tau.$$

Then, $\varphi_1(1) = 0$ and $\varphi_2(0) = 0$. Since $K_0(t,\tau)$ is continuous $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ are continuously differentiable. We obtain

$$\begin{aligned} \lambda\varphi_1'(t) &= \lambda A_{11}\varphi_1(t) + \lambda A_{12}\varphi_2(t) \\ \varphi_2'(t) &= -\frac{1}{\lambda}A_{21}\varphi_1(t) + A_{22}\varphi_2(t). \end{aligned}$$

Thus, $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ solve (23). Conversely, let $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ be solutions to the boundary value problem. Then,

$$\varphi_1(t) = -\int_t^1 e^{(t-\tau)A_{11}} A_{12}\varphi_2(\tau)\, d\tau \tag{24}$$

$$\varphi_2(t) = -\frac{1}{\lambda}\int_0^t e^{(t-\tau)A_{22}} A_{21}\varphi_1(\tau)\, d\tau. \tag{25}$$

Inserting (25) into (24) yields the eigenvalue equation (22). □

The solution theory of (23) is intimately related with the section determinant $\det U_{11}(1, -\frac{1}{\lambda})$.

Lemma 3.4. *The boundary value problem* (23) *possesses a non-trivial solution if and only if* $\det U_{11}(1, -\frac{1}{\lambda}) = 0$.

Proof. [4], Theorem 3.4. □

The following lemma is a simple consequence of the preceding results.

Lemma 3.5. $\lambda \neq 0$ *is an eigenvalue of* $\hat{K}_0$ *if and only if* $\det U_{11}(1, -\frac{1}{\lambda}) = 0$.

Proof. Combine Lemmas 3.3 and 3.4. □

We thus arrive at the operator version of Proposition 2.2.

Theorem 3.6. *Let* $\det U_{11}(1, \alpha) \neq 0$ *for all* $\alpha \in [0, 1]$. *Then the operator* $(\mathbb{1} + \alpha\hat{K}_0)^{-1}\hat{K}_0$ *is well-defined and trace class. Moreover, for the section determinant we have the formula*

$$\det U_{11}(1) = e^{\operatorname{tr} A_{11}} \exp\left[\int_0^1 \operatorname{tr}(\mathbb{1} + \alpha\hat{K}_0)^{-1}\hat{K}_0\, d\alpha\right]. \tag{26}$$

Proof. We know from Proposition 2.2 that there is at least one solution to the equation (18) respectively (21) such that formula (19) holds. By Lemma 3.5 $\det U_{11}(1, \alpha) \neq 0$ implies that $(\mathbb{1} + \alpha\hat{K}_0)^{-1}$ exists and is bounded. Hence, we can solve for $\hat{K}(\alpha)$ in (21) and deduce from Lemma 3.1 that $(\mathbb{1} + \alpha\hat{K}_0)^{-1}\hat{K}_0$ is trace class. The statement follows directly from Proposition 2.2. □

We recall the formula

$$\frac{d}{d\alpha}\ln\det(\mathbb{1} + \alpha S) = \operatorname{tr}(\mathbb{1} + \alpha S)^{-1} S, \tag{27}$$

which is valid whenever $(\mathbb{1} + \alpha S)^{-1}$ exists and S is trace class (see [3], Chap. IV, (1.14)). By the way we want to mention that this formula also provides the ground for Proposition 2.1. By reading (27) from the right to the left we can derive the main result of this paper as simple corollary of the above theorem.

Corollary 3.7. *Let* $\det U_{11}(1,\alpha) \neq 0$ *for all* $\alpha \in [0,1]$. *Then the section determinant* $\det U_{11}(1)$ *essentially equals the Fredholm determinant of the integral operator* $\hat{K}_0$ *having the kernel function* (17)*:*

$$\det U_{11}(1) = e^{\operatorname{tr} A_{11}} \det(\mathbb{1} + \hat{K}_0). \tag{28}$$

Proof. Insert formula (27) into formula (26). □

A further simple consequence is a product representation of $\det U_{11}(1)$ in terms of the zeroes of $\det U_{11}(1,\varkappa)$.

Corollary 3.8. *The section determinant can be represented by*

$$\det U_{11}(1) = e^{\operatorname{tr} A_{11}} \prod_{j=1}^{\infty} \left(1 - \frac{1}{\varkappa_j}\right) \tag{29}$$

with $\varkappa_j \neq 0$ *being the zeroes of* $\det U_{11}(1,\varkappa)$ *with respect to* $\varkappa$.

Proof. It is clear from (9) that $\varkappa = 0$ cannot be a zero of $\det U_{11}(1,\varkappa)$. Recalling the properties of the Fredholm determinant (see e.g. [3]) we deduce the statement from Corollary 3.7 and Lemma 3.5. □

The product in (29) is of Weierstrass type. This observation provides an alternative approach to our main result (28). To be more precise, one can show that the function $\varkappa \to \det U_{11}(1,\varkappa)$ is an entire holomorphic function with respect to $\varkappa$ and has at most exponential growth. Then Corollary 3.8 follows from Hadamard's factorization theorem and we can derive (28) from (29) with the aid of Lemma 3.5. Whatever way is chosen to prove (28) it relies heavily on the special decomposition (5) of A. The method presented here has the advantage that it is Theorem 3.6 rather than its corollaries that may prove itself useful in studying section determinants from a more general and abstract point of view.

4. A criterion

Thus far we have not derived any statement concerning the actual location of the spectrum of $\hat{K}_0$. This will be done now for self-adjoint A. We first prove a general result.

Lemma 4.1. *Let* $S : \mathcal{H} \to \mathcal{H}$ *be a bounded self-adjoint operator and* $V(t) := e^{tS}$. *Then* $\det V_{11}(t) \neq 0$ *for all* $t \in \mathbb{R}$.

Proof. We show that $V_{11}(t)$ is invertible for all $t \in \mathbb{R}$. Since S is self-adjoint so is $V(t)$ which implies $V_{jk}(t)^* = V_{kj}(t)$ for $j,k = 1,2$. Recalling the group property $V(t) = V(t/2)V(t/2)$ yields

$$V_{11}(t) = V_{11}^2(t/2) + V_{12}(t/2)V_{21}(t/2). \tag{30}$$

Assume that $V_{11}(t)\varphi = 0$ for some $\varphi \in \mathcal{H}_1$. Then it follows from (30) that $V_{11}(t/2)\varphi = 0$ because of the right-hand side being positive. Repeating this argument yields $V_{11}(t/2^n)\varphi = 0$. Now we take the limit $n \to \infty$ to conclude

$$0 = \lim_{n\to\infty} V_{11}(t/2^n)\varphi = \varphi$$

where we used the continuity of $V(t)$ with respect to t. This proves the lemma. $\square$

There are several ways of proving the above statement. The one presented carries over to the case when S is just self-adjoint without necessarily being bounded because then $V(t)$ becomes a strongly continuous semi-group which is still sufficient to make the proof work. We cannot apply the result immediately to our problem of proving $\det U_{11}(1,\alpha) \neq 0$ because $A(\alpha)$ is not self-adjoint. Nonetheless, we can show that $A(\alpha)$ is similar to a self-adjoint operator that generates the same section determinant.

Lemma 4.2. *Let A be self-adjoint. Then $\det U_{11}(1,\varkappa) \neq 0$ for all $\varkappa \geq 0$.*

Proof. The case $\varkappa = 0$ is clear from (9). Let $\varkappa > 0$ and define

$$J(\varkappa) := \begin{pmatrix} \sqrt{\varkappa}\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}.$$

Then,

$$\tilde{A}(\varkappa) := J(\varkappa)A(\varkappa)J^{-1}(\varkappa) = \begin{pmatrix} A_{11} & \sqrt{\varkappa}A_{12} \\ \sqrt{\varkappa}A_{21} & A_{22} \end{pmatrix}$$

is self-adjoint because $\sqrt{\varkappa} \in \mathbb{R}$. Let $\tilde{U}(t,\varkappa) := e^{t\tilde{A}(\varkappa)}$. By dint of $\tilde{U}(1,\varkappa) = W(\varkappa)U(1,\varkappa)W^{-1}(\varkappa)$ and Lemma 4.1 follows

$$\det U_{11}(1,\varkappa) = \det \tilde{U}_{11}(1,\varkappa) \neq 0.$$

This proves the lemma. $\square$

Having found an concrete criterion on the resolvent set of $\hat{K}_0$ we can formulate Theorem 3.6 and its Corollary 3.7 for self-adjoint A without any additional restriction.

Theorem 4.3. *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded self-adjoint operator and $U(t) := e^{tA}$. Then,*

$$\det U_{11}(1) = e^{\operatorname{tr} A_{11}} \exp\left[\int_0^1 \operatorname{tr}(\mathbb{1} + \alpha\hat{K}_0)^{-1}\hat{K}_0\, d\alpha\right] \tag{31}$$

and

$$\det U_{11}(1) = e^{\operatorname{tr} A_{11}} \det(\mathbb{1} + \hat{K}_0). \tag{32}$$

Proof. By Lemma 4.2 we have $\det U_{11}(1,\alpha) \neq 0$. Thus the conditions of Theorem 3.6 are satisfied. The formulae follow immediately from Theorem 3.6 and Corollary 3.7. $\square$

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