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ENCLOSURE RESULTS FOR SECOND ORDER RELATIVE SPECTRA BY ELEMENTARY MEANS

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ABSTRACT. Motivated by the general approach due to Shargorodsky we derive enclosure results for the second order relative spectrum of bounded self-adjoint operators by studying quadratic operator pencils. The quality of the results is discussed by means of a simple example.

Higher order relative spectra of self-adjoint operators were introduced by Davies [1], who used particularly second order spectra as a new definition for resonances (see Howland [2]). Subsequently, Shargorodsky [5] obtained general enclosure results for the n-th order spectrum.

Here we focus on second order spectra whereby completely elementary tools from Hilbert space theory suffice not only to rediscover but also to strengthen some of the enclosure results. The key idea is to write the operator being considered as operator block matrix and thus to link the question of second order spectra with the invertibility of some special quadratic operator pencils.

Let $\mathcal{H}$ be a complex Hilbert space, $A, B : \mathcal{H} \to \mathcal{H}$ bounded self-adjoint operators, and $\mathbb{1} : \mathcal{H} \to \mathcal{H}$ the identity. We consider the quadratic operator pencil

$$M(z) := (A - z\mathbb{1})^2 + B^2,\ z \in \mathbb{C},$$

and define

$$\rho(A, B) := \{z \in \mathbb{C} | M(z) \text{ has a bounded inverse}\}$$

and $\sigma(A, B) := \mathbb{C} \setminus \rho(A, B)$. These resemble the ordinary resolvent set and spectrum. We introduce

$$\begin{array}{lll} \alpha_{min} := \inf \sigma(A) & \beta_{min} := (\inf \sigma(B^2))^{1/2} & \gamma_{min} := (\inf \sigma(A^2 + B^2))^{1/2} \\ \alpha_{max} := \sup \sigma(A) & \beta_{max} := (\sup \sigma(B^2))^{1/2} & \gamma_{max} := (\sup \sigma(A^2 + B^2))^{1/2} \end{array}$$

with σ denoting the (usual) spectrum of the respective operator.

Theorem 1. *Let $z \in \sigma(A, B)$ with $z = x + iy$, $x, y \in \mathbb{R}$. Then,*

(i) $x \in [\alpha_{min}, \alpha_{max}]$. *If $y = 0$ then $x \in \sigma(A) \subset [\alpha_{min}, \alpha_{max}]$.*
(ii) $|y| \in [\beta_{min}, \beta_{max}]$.
(iii) $\gamma_{min} \leq |z| \leq \gamma_{max}$.

Proof. To show invertibility of $M(z)$ we invoke the Lax-Milgram Theorem (see e.g. [4]) if not indicated otherwise.

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i) Using the triangle inequality we obtain

$$|\operatorname{Im}(\varphi, M(z)\varphi)| \geq 2|y|(x-\alpha_{max})\|\varphi\|^2 \quad \text{and} \quad |\operatorname{Im}(\varphi, M(z)\varphi)| \geq 2|y|(\alpha_{min}-x)\|\varphi\|^2$$

which proves the case $y \neq 0$ and $x \notin [\alpha_{min}, \alpha_{max}]$.

If $y = 0$ and $x \notin \sigma(A)$ the statement is implied by

$$\operatorname{Re}(\varphi, M(z)\varphi) = \|(A - x\mathbb{1})\varphi\|^2 + (\varphi, B^2\varphi) \geq \frac{1}{\|(A - x\mathbb{1})^{-1}\|^2}\|\varphi\|^2.$$

ii) The left boundary follows from

$$\operatorname{Re}(\varphi, M(z)\varphi) = \|(A - x\mathbb{1})\varphi\|^2 + (\varphi, B^2\varphi) - y^2 \geq (\beta_{min}^2 - y^2)\|\varphi\|^2.$$

If $|y| > \beta_{max}$ we write

$$M(z) = (A - z\mathbb{1})^2\left[\mathbb{1} + ((A - z\mathbb{1})^{-1})^2 B^2\right].$$

The statement is then proved with the aid of the norm estimate for the resolvent (see e.g. [3]), which yields

$$\|((A - z\mathbb{1})^{-1})^2 B^2\| \leq \frac{\beta_{max}^2}{y^2}.$$

iii) If $y = 0$ then $x \in \sigma(A)$ by i) and hence $|x| \leq \max\{|\alpha_{min}|, |\alpha_{max}|\} \leq \gamma_{max}$.

Let $y \neq 0$. Then we may eliminate $(\varphi, A\varphi)$ from the real part by dint of the imaginary part and, having rearranged and taken absolute values, obtain

$$|(x^2 + y^2)\|\varphi\|^2 - (\varphi, (A^2 + B^2)\varphi)| = |-\operatorname{Re}(\varphi, M(z)\varphi) + \frac{x}{y}\operatorname{Im}(\varphi, M(z)\varphi)|.$$

Applying the Cauchy-Schwarz inequality to the right-hand side and estimating the left-hand side from below yield

$$C\|\varphi\|^2 \leq (1 + \frac{x^2}{y^2})^{1/2}|(\varphi, M(z)\varphi)|$$

with either $C = x^2 + y^2 - \gamma_{max}^2$ or $C = \gamma_{min}^2 - x^2 - y^2$ which proves the statement. $\square$

We discuss the above bounds. For $\alpha := \max\{|\alpha_{min}|, |\alpha_{max}|\}$ we have $0 \leq \alpha \leq \gamma_{max}$. Furthermore, $0 \leq \beta_{max} \leq \gamma_{max}$. Note $\beta_{min} \leq \beta_{max}$ because $B^2 \geq 0$. In the case $\alpha^2 + \beta_{max}^2 \leq \gamma_{max}^2$ combining bounds i) and ii) gives a sharper result than iii). If $\alpha^2 + \beta_{max}^2 > \gamma_{max}^2$ all three bounds i) through iii) must be used in order to yield the best estimates. The interested reader may draw some pictures by himself to visualize these results.

Thus far we have disregarded the relationship between A and B, e.g. commutation properties. One could hope for more accurate estimates by using such properties. We leave this out for future investigations.

For a bounded self-adjoint operator $H : \mathcal{H} \to \mathcal{H}$ and an orthogonal projection $P : \mathcal{H} \to \mathcal{H}$ Davies [1] defines the n-th order relative resolvent set and spectrum of H by

$$\rho_n(H, P) := \{z \in \mathbb{C} | P(H - z\mathbb{1})^n P : \operatorname{ran} P \to \operatorname{ran} P \text{ has a bounded inverse}\}$$

and $\sigma_n(H, P) := \mathbb{C} \setminus \rho_n(H, P)$, respectively. We write $\mathcal{H}$ as orthogonal sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with $\mathcal{H}_1 := \operatorname{ran} P$ and $\mathcal{H}_2 := \mathcal{H}_1^\perp$ and H as 2×2 block matrix:

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \ H_{jk} : \mathcal{H}_k \to \mathcal{H}_j, \ j, k = 1, 2.$$

It is easily seen that the H_{jk} are bounded and obey $H_{jk}^* = H_{kj}$. In the case $n = 2$

$$M(z) := P(H - z\mathbb{1})^2 P = (H_{11} - z\mathbb{1})^2 + H_{12}H_{21} : \mathcal{H}_1 \to \mathcal{H}_1.$$

This is a quadratic operator pencil with $H_{12}H_{21} \geq 0$ wherefore we can apply Theorem 1 by putting $A := H_{11}$ and $B^2 := H_{12}H_{21}$. We do this for iii). The accordant spectral quantity, γ_{max}, can be estimated further

$$\gamma_{max}^2 = \sup \sigma(H_{11}^2 + H_{12}H_{21}) = \sup_{P\psi \neq 0} \frac{(P\psi, H^2 P\psi)}{\|P\psi\|^2} \leq \|H\|^2.$$

We then deduce immediately

Theorem 2. *Let $H : \mathcal{H} \to \mathcal{H}$ be a bounded self-adjoint operator. Then,*

$$\sigma_2(H, P) \subset \{z \in \mathbb{C} |\ |z| \leq \gamma_{max}\} \subset \{z \in \mathbb{C} |\ |z| \leq \|H\|\}.$$

As show simple examples of 3×3 diagonal matrices, in general, the strict inequality $\gamma_{max} < \|H\|$ holds, whence the first inclusion in Theorem 2 provides more accurate information on $\sigma_2(H, P)$ than the second one, which is a special case of a theorem of Shargorodsky [5].

The larger circle in Theorem 2 is optimal if the spectrum of H is symmetric with respect to the real axis which we can always achieve by shifting the spectrum.

Corollary 3. *Let $H : \mathcal{H} \to \mathcal{H}$ be bounded and self-adjoint and*

$$\delta_{min} := \inf \sigma(H), \ \ \delta_{max} := \sup \sigma(H).$$

Then,

$$\sigma_2(H, P) \subset \{z \in \mathbb{C} |\ |z - \frac{1}{2}(\delta_{min} + \delta_{max})| \leq \frac{1}{2}(\delta_{max} - \delta_{min})\}.$$

Proof. Apply Theorem 2 to the operator $\tilde{H} := H - \frac{1}{2}(\delta_{min} + \delta_{max})\mathbb{1}$. □

Corollary 3 is best possible under the assumptions given in that the radius of that circle cannot be lessened and a circle is the only possible curve that can enter in any theorem of this type. To see this consider

$$H(\lambda, V) := \begin{pmatrix} \lambda & V \\ \bar{V} & -\lambda \end{pmatrix}, \ \lambda \in \mathbb{R}, \ V \in \mathbb{C}, \quad P := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

$H(\lambda, V)$ has eigenvalues $\pm\sqrt{\lambda^2 + |V|^2}$ and, consequently, $\|H(\lambda, V)\| = \sqrt{\lambda^2 + |V|^2}$. Finally, $M(z) = (\lambda - z)^2 + |V|^2$, which is invertible if and only if $M(z) \neq 0$, whence,

$$\sigma_2(H(\lambda, V), P) = \{\lambda \pm i|V|\}.$$

If $z \in \sigma_2(H(\lambda, V), P)$ we have $|z| = \sqrt{\lambda^2 + |V|^2} = \|H\|$ and $x = \lambda$. For $\|H\|$ fixed and $V := \sqrt{\|H\|^2 - \lambda^2}$ the circle in Corollary 3 is the same for all $\lambda \in [-\|H\|, \|H\|]$ and $\sigma_2(H(\lambda, V), P)$ runs through the whole circle while λ runs from $-\|H\|$ to $\|H\|$.

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