Second Order Spectrum of Bounded Self-Adjoint Operators

Peter Otte

October 23, 1998

Abstract

The spectrum of quadratic operator pencils is studied. The results are applied to the second order spectrum of bounded self-adjoint operators and discussed by means of a simple example.

1 Introduction

In [1] Davies introduced the notion of higher order spectra of a self-adjoint operator H with respect to an orthogonal projection P. These are those points such that $P(H - z\mathbb{1})^n P$ is not invertible on $\operatorname{ran} P$. The first order spectrum obviously coincides with the ordinary spectrum of $H\,|_{\operatorname{ran} P}$ and thus there is no new information if $P = \mathbb{1}$. The second order spectrum is used by Davies as a new definition for resonances (see Howland [2]).

Leaving aside the trivial case $P = \mathbb{1}$ a self-adjoint operator may have non-real spectral values, what makes locating those spectral values in the complex plane an interesting task. In order to study the second order spectrum in detail, it is appropriate to write the underlying Hilbert space as orthogonal sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ whence the operators under consideration have a 2×2 block matrix form. Then studying the second order spectrum reduces to investigating the invertibility of some special quadratic operator pencils $(A - z\mathbb{1})^2 + B^2$. This is done in Section 2. We derive some enclosure results for σ which involve spectral data of A and B but do not take into account the relationship between A and B what would provide, probably, more accurate estimates. We leave this out for future investigations.

The results proved for operator pencils easily carry over to the second order spectrum of a bounded self-adjoint operator by simply changing the notation. Here neglecting the relation between A and B means suppressing the interplay of H and P. As was the case before, this would give more information about the shape of the second order spectrum. On the other hand, one can completely ignore the projection P because the results can be made independent of it (Theorem 3.3). Corollary 3.4 strengthens a statement of Davies [1] and is the strongest possible under the given assumptions as is seen by means of a simple example (Section 4).

Related results concerning second order spectra were obtained independently by Shargorodsky [5].

2 Quadratic operator pencils

Let A and B be bounded self-adjoint operators acting on a complex Hilbert space $\mathcal{H}$. We consider the quadratic operator pencil:

$$M(z) := (A - z\mathbb{1})^2 + B^2, \ z \in \mathbb{C} \tag{1}$$

and ask for those z for which $M(z)$ is invertible. Here $\mathbb{1} : \mathcal{H} \to \mathcal{H}$ denotes the identity operator. Because of the appearance of B^2 rather than B we restrict ourselves to positive operators. This is sufficient in view of Section 3.

Definition 2.1. *Let $M(z)$ be as in (1). We call*

$$\rho(A, B) := \{z \in \mathbb{C} | M(z) \text{ has a bounded inverse}\}$$

the resolvent set of $M(z)$ and $\sigma(A, B) := \mathbb{C} \setminus \rho(A, B)$ the spectrum of $M(z)$.

In some sense $M(z)^{-1}$ generalizes the usual resolvent. Thus $\sigma(A, B)$ and $\rho(A, B)$ may be considered generalizations of the spectrum and the resolvent set of a single operator, what justifies calling them also spectrum and resolvent set. We want to describe how $\sigma(A, B)$, respectively $\rho(A, B)$, is located in the complex plane. To this end we study the quadratic form $(\varphi, M(z)\varphi)$. We compute the real part and imaginary part. In the sequel we always write $z = x + iy$, $x, y \in \mathbb{R}$ without further mentioning.

Lemma 2.2. *Let $M(z)$ be as in (1). Then for all $\varphi \in \mathcal{H}$:*

$$\mathrm{Re}(\varphi, M(z)\varphi) = (\varphi, (A - x\mathbb{1})^2\varphi) + (\varphi, B\varphi) - y^2\|\varphi\|^2, \tag{2}$$

$$\mathrm{Im}(\varphi, M(z)\varphi) = 2y(-(\varphi, A\varphi) + x\|\varphi\|^2). \tag{3}$$

Proof. Straightforward calculations. □

We first consider the real part of $z \in \sigma(A, B)$. It turns out that it is governed only by the spectral bounds of A.

Theorem 2.3. *Let $z \in \sigma(A, B)$. Then $x \in [\alpha_{min}, \alpha_{max}]$ where we have put:*

$$\alpha_{min} := \inf_{\|\varphi\|=1} (\varphi, A\varphi), \ \ \alpha_{max} := \sup_{\|\varphi\|=1} (\varphi, A\varphi).$$

Moreover, if $y = 0$ then $x \in \sigma(A)$. Here $\sigma(A)$ is the spectrum of A.

Proof. Let us start with $y \neq 0$. We use (3):

$$|\,\mathrm{Im}(\varphi, M(z)\varphi)| \geq 2|y|(x\|\varphi\|^2 - (\varphi, A\varphi)) \geq 2|y|(x - \alpha_{max})\|\varphi\|^2$$

and

$$|\,\mathrm{Im}(\varphi, M(z)\varphi)| \geq 2|y|((\varphi, A\varphi) - x\|\varphi\|^2) \geq 2|y|(\alpha_{min} - x)\|\varphi\|^2.$$

Hence the Lax-Milgram Theorem (see Theorem A.1 in the appendix) yields the invertibility of $M(z)$ for $x \notin [\alpha_{min}, \alpha_{max}]$.

If $y = 0$ we look at (2):

$$\mathrm{Re}(\varphi, M(z)\varphi) = \|(A - x\mathbb{1})\varphi\|^2 + (\varphi, B^2\varphi) \geq \|(A - x\mathbb{1})\varphi\|^2.$$

Let $x \notin \sigma(A)$. Then $(A - x\mathbb{1})^{-1}$ exists and is bounded by definition. We have:

$$\|\varphi\| = \|(A - x\mathbb{1})^{-1}(A - x\mathbb{1})\varphi\| \leq \|(A - x\mathbb{1})^{-1}\|\|(A - x\mathbb{1})\varphi\|.$$

This yields the estimate:

$$\|(A - x\mathbb{1})\varphi\| \geq \frac{1}{\|(A - x\mathbb{1})^{-1}\|}\|\varphi\|.$$

Hence Theorem A.1 shows that $M(z)$ is invertible. $\square$

We turn to the imaginary part. As one expects there are only non-real values in $\sigma(A, B)$ if $B \neq 0$. This can be made more precise by giving an estimate on y depending on B in such a way that y tends to 0 as B does. On the other hand, if B^2 is strictly positive then there are no real values in $\sigma(A, B)$ at all.

Theorem 2.4. *Let $z \in \sigma(A, B)$ and let $\beta_{min}, \beta_{max} \geq 0$ be defined by:*

$$\beta_{min}^2 := \inf_{\|\varphi\|=1} (\varphi, B^2\varphi), \ \beta_{max}^2 := \sup_{\|\varphi\|=1} (\varphi, B^2\varphi).$$

Then:

$$\beta_{min} \leq |y| \leq \beta_{max}.$$

Proof. In order to show the first inequality we employ (2):

$$\mathrm{Re}(\varphi, M(z)\varphi) = \|(A - x\mathbb{1})\varphi\|^2 + (\varphi, B^2\varphi) - y^2 \geq (\beta_{min}^2 - y^2)\|\varphi\|^2.$$

We conclude by the aid of Theorem A.1 that $M(z)$ is invertible if $|y| < \beta_{min}$.

We prove the second inequality. Let $y \neq 0$. Then the resolvent $(A - z\mathbb{1})^{-1}$ exists as a bounded operator with

$$\|(A - z\mathbb{1})^{-1}\| \leq \frac{1}{|y|}$$

(see e.g. [3]). Hence we may rewrite $M(z)$:

$$M(z) = (A - z\mathbb{1})^2 \left[\mathbb{1} + ((A - z\mathbb{1})^{-1})^2 B^2\right].$$

We have the estimate

$$\|((A - z\mathbb{1})^{-1})^2 B^2\| \leq \frac{\beta_{max}^2}{y^2}.$$

If $|y| > \beta_{max}$ the Neumann series (see [4]) shows that $\mathbb{1} + ((A - z\mathbb{1})^{-1})^2 B^2$ is invertible, thus implying the existence and boundedness of $M(z)^{-1}$. $\square$

So far we have considered the real part and the imaginary part separately. Now we relate them to each other by using both formula (2) and formula (3).

Theorem 2.5. *Let $z \in \sigma(A, B)$. Then*

$$\gamma_{min} \leq |z| \leq \gamma_{max}$$

where $\gamma_{min} \geq 0$ and $\gamma_{max} \geq 0$ are given by:

$$\gamma_{min}^2 := \inf_{\|\varphi\|=1} (\varphi, (A^2 + B^2)\varphi), \ \gamma_{max}^2 := \sup_{\|\varphi\|=1} (\varphi, (A^2 + B^2)\varphi).$$

Proof. If $y = 0$ then $x \in \sigma(A)$ by Theorem 2.3 and hence $|x| \leq \alpha$ with $\alpha := \max\{|\alpha_{min}|, |\alpha_{max}|\}$. Now it is easily seen that $\alpha \leq \gamma_{max}$ what proves the statement in this case.

Let $y \neq 0$. Then we may solve for $(\varphi, A\varphi)$ in (3) and substitute this into (2) hence obtaining:

$$\begin{aligned} \operatorname{Re}(\varphi, M(z)\varphi) &= (\varphi, (A^2 + B^2)\varphi) + \frac{x}{y} \operatorname{Im}(\varphi, M(z)\varphi) - 2x^2\|\varphi\|^2 + (x^2 - y^2)\|\varphi\|^2 \\ &= (\varphi, (A^2 + B^2)\varphi) + \frac{x}{y} \operatorname{Im}(\varphi, M(z)\varphi) - (x^2 + y^2)\|\varphi\|^2. \end{aligned}$$

Rearranging yields:

$$(x^2 + y^2)\|\varphi\|^2 - (\varphi, (A^2 + B^2)\varphi) = -\operatorname{Re}(\varphi, M(z)\varphi) + \frac{x}{y} \operatorname{Im}(\varphi, M(z)\varphi).$$

Now we are able to estimate by using the Cauchy-Schwarz inequality on the right-hand side:

$$\begin{aligned} \left[(x^2 + y^2) - \gamma_{max}^2\right] \|\varphi\|^2 &\leq (1 + \frac{x^2}{y^2})^{1/2} (\operatorname{Re}^2(\varphi, M(z)\varphi) + \operatorname{Im}^2(\varphi, M(z)\varphi))^{1/2} \\ &= (1 + \frac{x^2}{y^2})^{1/2} |(\varphi, M(z)\varphi)|. \end{aligned}$$

Analogously we obtain:

$$\left[\gamma_{min}^2 - (x^2 + y^2)\right] \leq (1 + \frac{x^2}{y^2})^{1/2} |(\varphi, M(z)\varphi)|.$$

Theorem A.1 proves the invertibility of $M(z)$ if $x^2 + y^2 > \gamma_{max}^2$ or $x^2 + y^2 < \gamma_{min}^2$. □

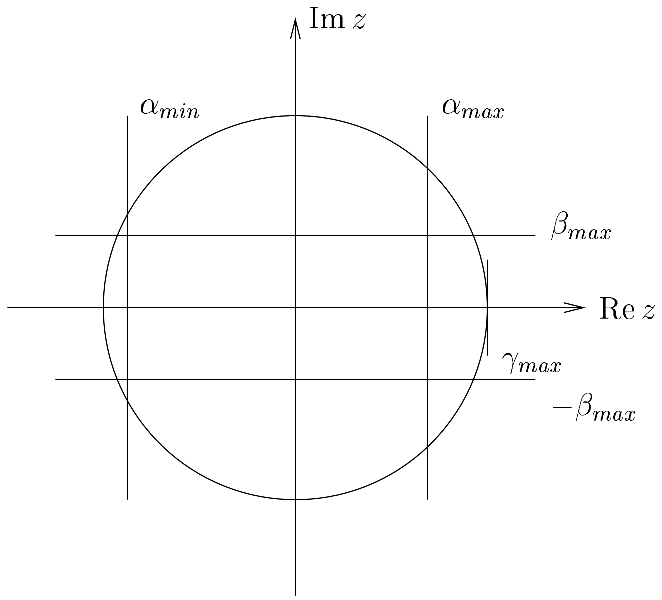
We want to illustrate how the preceding theorems are related to each other and, in particular, which combination of them yields optimal results. Therefore we compare the spectral bounds of A and B. As was already noted in the proof of Theorem 2.5 we have:

$$0 \leq \alpha \leq \gamma_{max}.$$

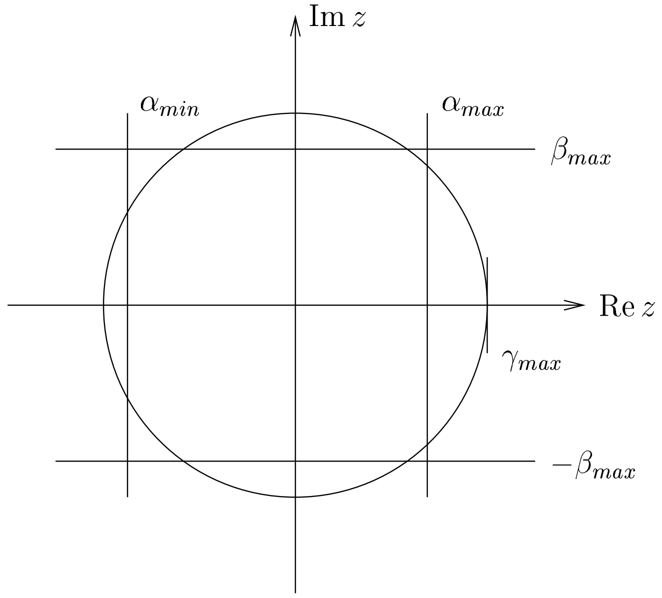
The same is true for β_{max}:

$$0 \leq \beta_{max} \leq \gamma_{max}.$$

Note that $\beta_{min} \leq \beta_{max}$ since their definition involved the positive operator B^2 instead of B. Now it may occur that $\alpha^2 + \beta_{max}^2 \leq \gamma_{max}^2$. Then combining Theorem 2.3 with Theorem 2.4 gives a sharper result than Theorem 2.5. This situation is displayed by the following picture:



Otherwise, if $\alpha^2+\beta_{max}^2 > \gamma_{max}^2$ then all three Theorems 2.3 through 2.5 must be used together in order to optimal estimate the shape of $\sigma(A, B)$. This is shown by the picture below.



3 Second order spectrum

Let $\mathcal{H}$ be a complex Hilbert space and $P : \mathcal{H} \to \mathcal{H}$ an orthogonal projection. Following Davies [1] we define the higher order resolvent set and spectrum of a self-adjoint operator relative to P.

Definition 3.1. *Let $H : \mathcal{H} \to \mathcal{H}$ be a bounded self-adjoint operator. Then the n-th order resolvent set of H relative to P is defined as:*

$$\rho_n(H, P) := \{z \in \mathbb{C} | P(H - z\mathbb{1})^n P : \operatorname{ran} P \to \operatorname{ran} P \textit{ has a bounded inverse}\}.$$

We call $\sigma_n(H, P) := \mathbb{C} \setminus \rho_n(H, P)$ the n-th order spectrum of H relative to P.

For $P = \mathbb{1}$ and $n = 1$ we recover the ordinary spectrum. Here we focus on the case $n = 2$ and arbitrary P. We write $\mathcal{H}$ as an orthogonal sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with $\mathcal{H}_1 := \operatorname{ran} P$ and $\mathcal{H}_2 := \mathcal{H}_1^\perp$ and H as 2×2 block matrix:

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \; H_{jk} : \mathcal{H}_k \to \mathcal{H}_j, \; j, k = 1, 2. \tag{4}$$

Of course, the operators H_{jk} are bounded since H is. We write $P(H - z\mathbb{1})^2 P$ in terms of these operators:

$$M(z) := (H_{11} - z\mathbb{1})^2 + H_{12}H_{21} : \mathcal{H}_1 \to \mathcal{H}_1. \tag{5}$$

This is a quadratic operator pencil in the sense of Section 2 since obviously $H_{12}H_{21} \geq 0$. Therefore all results carry over by putting $A := H_{11}$ and $B^2 := H_{12}H_{21}$. We do not explicitely cast all of the theorems of the foregoing section into the new form but restrict ourselves to Theorem 2.5.

Theorem 3.2. *Let $H : \mathcal{H} \to \mathcal{H}$ be a bounded self-adjoint operator which is written as block matrix according to* (4)*. Then*

$$\sigma_2(H, P) \subset \{z \in \mathbb{C} | \; |z| \leq \gamma\}$$

with $\gamma \geq 0$ being defined as

$$\gamma^2 := \sup_{\|\varphi\|=1} (\varphi, (H_{11}^2 + H_{12}H_{21})\varphi).$$

Proof. Put $A = H_{11}$ and $B^2 = H_{12}H_{21}$ in Theorem 2.5. □

The last theorem can be made independent of P.

Theorem 3.3. *Let $H : \mathcal{H} \to \mathcal{H}$ be a bounded self-adjoint operator. Then*

$$\sigma_2(H, P) \subset \{z \in \mathbb{C} | |z| \leq \|H\|\}.$$

Proof. We estimate γ in Theorem 3.2:

$$\begin{aligned} \gamma^2 &= \sup_{\|\varphi\|=1} (\varphi, (H_{11}^2 + H_{12}H_{21})\varphi) \\ &= \sup_{P\psi \neq 0} \frac{(P\psi, (H_{11}^2 + H_{12}H_{21})P\psi)}{\|P\psi\|^2} \\ &= \sup_{P\psi \neq 0} \frac{(P\psi, H^2 P\psi)}{\|P\psi\|^2} \\ &\leq \sup_{\psi \neq 0} \frac{(\psi, H^2\psi)}{\|\psi\|^2} \\ &= \|H\|^2. \end{aligned}$$

Hence the statement follows. □

We note that the circle in Theorem 3.3 is optimal if the spectrum of H is symmetric with respect to the real axis. But this can always be achieved by simply shifting the spectrum.

Corollary 3.4. *Let $H : \mathcal{H} \to \mathcal{H}$ be self-adjoint. Let*

$$\delta_{min} := \inf\{z \in \sigma(H)\}, \ \delta_{max} := \sup\{z \in \sigma(H)\}.$$

Then

$$\sigma_2(H, P) \subset \{z \in \mathbb{C} | |z - \frac{\delta_{min} + \delta_{max}}{2}| \leq \frac{\delta_{max} - \delta_{min}}{2}\}.$$

Proof. We apply Theorem 3.3 to the operator $\tilde{H} := H - \frac{\delta_{min}+\delta_{max}}{2}\mathbb{1}$:

$$\sigma_2(\tilde{H}, P) \subset \{z \in \mathbb{C} | |z| \leq \|\tilde{H}\|\}.$$

Since $P(H - (z + \frac{\delta_{min}+\delta_{max}}{2})\mathbb{1})^n P$ is invertible on $\operatorname{ran} P$ if and only if $P(\tilde{H} - z\mathbb{1})^n P$ is we conclude $z \in \sigma_2(\tilde{H}, P)$ if and only if $z + \frac{\delta_{min}+\delta_{max}}{2} \in \sigma_2(H, P)$. Then the statement follows from $\|\tilde{H}\| = \frac{\delta_{max}-\delta_{min}}{2}$. □

The above corollary strenghtens a result derived by Davies [1]. We discuss its optimality in the next section.

4 An Example

We show that Corollary 3.4 is optimal under the given assumptions in that the radius of the circle cannot be chosen smaller and the circle is the only possible curve which can enter in any theorem of this type. Consider the matrices

$$H(\lambda, V) := \begin{pmatrix} \lambda & V \\ \bar{V} & -\lambda \end{pmatrix}, \ \lambda \in \mathbb{R}, \ V \in \mathbb{C}. \tag{6}$$

The eigenvalues of $H(\lambda, V)$ are $\pm\sqrt{\lambda^2 + |V|^2}$. Thus the norm of $H(\lambda, V)$ is easily seen to be $\|H(\lambda, V)\| = \sqrt{\lambda^2 + |V|^2}$. We choose the orthogonal projection

$$P := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{7}$$

Then $M(z)$ in (5) becomes:

$$M(z) = (\lambda - z)^2 + |V|^2,$$

which is invertible if and only if $M(z) \neq 0$. Hence we can immediately compute the second order spectrum of H:

$$\sigma_2(H(\lambda, V), P) = \{\lambda \pm i|V|\}.$$

For $z \in \sigma_2(H(\lambda, V), P)$ we have:

$$|z| = \sqrt{\lambda^2 + |V|^2} = \|H\|, \ \operatorname{Re} z = \lambda.$$

Now let $\|H\|$ be fixed and put $V := \sqrt{\|H\|^2 - \lambda^2}$. Then the circle in Corollary 3.4 is the same for all $\lambda \in [-\|H\|, \|H\|]$ and $\sigma_2(H(\lambda, V), P)$ runs through the whole circle when λ runs from $-\|H\|$ to $\|H\|$, thus showing that no other curve can be chosen in Corollary 3.4.

A The Lax-Milgram Theorem

For the sake of the reader's convenience we cite the Lax-Milgram Theorem, which is the basis of most of our proofs in the preceding sections.

Theorem A.1. *Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. If there is a $\tau > 0$ such that for all $\varphi \in \mathcal{H}$*

$$|(\varphi, T\varphi)| \geq \tau \|\varphi\|^2 \tag{8}$$

then T has a bounded inverse with $\|T^{-1}\| \leq \frac{1}{\tau}$.

A proof can be found in [4] where the condition

$$\mathrm{Re}(\varphi, T\varphi) \geq \tau \|\varphi\|^2$$

is used instead of (8). But it is easily seen that the proof also works in this case. We have used (8) because this covers all the different versions that appeared throughout the paper.

References

[1] Davies, E.B., Spectral enclosures and complex resonances for general self-adjoint operators , Preprint 1997

[2] Howland, J.S., The Livsic Matrix in Perturbation Theory, Journ. Math. Anal. Appl. **50**, 415-437 (1975)

[3] Kato, T., Perturbation theory for linear operators, Springer Verlag 1966

[4] Kress, R., Linear Integral Equations, Springer Verlag 1989

[5] Shargorodsky, E., Geometry of higher order relative spectra and projection methods. To appear in Journal of Operator Theory.