Section determinants of unitary operators and the operator-valued Riccati differential equation

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Abstract

A special initial value problem for the operator-valued Riccati differential equation (RDE) is used for studying section determinants of unitary operators. A solvability criterion for this initial value problem is derived by providing an a-priori estimate of the solution. Alternatively, the same estimate is derived by a method that is based entirely upon linear algebra.

1. *Introduction*

We examine section determinants of families of unitary operators $U(t)$, $t \in \mathbb{R}$, that appear as solutions to a Schrödinger equation:

$$i\frac{d}{dt}U(t) = AU(t),\ U(0) = \mathbb{1}. \tag{1·1}$$

Here A is a bounded self-adjoint operator acting on a Hilbert space $\mathcal{H}$ and $\mathbb{1}$ is the identity operator. Let $\mathcal{H}$ be written as orthogonal sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and represent $U(t)$ by a 2×2 block matrix:

$$U(t) = \begin{pmatrix} U_{11}(t) & U_{12}(t) \\ U_{21}(t) & U_{22}(t) \end{pmatrix}. \tag{1·2}$$

If $\mathcal{H}_1$ is of finite dimension then the section determinant $\det U_{11}(T)$ is well-defined. In order to study it, we use the following formula:

$$\det U_{11}(T) = e^{-iT\,\mathrm{tr}\,A_{11}} \exp\left[-i\int_0^T \mathrm{tr}\,A_{12}Y(t)\,dt\right]. \tag{1·3}$$

Here, $\mathrm{tr}\,S$ denotes the trace of operators S acting on $\mathcal{H}_1$ and A has been decomposed analogously to (1·2). The operator $Y(t) : \mathcal{H}_1 \to \mathcal{H}_2$ satisfies the following initial value problem for the Riccati differential equation (RDE):

$$i\frac{d}{dt}Y(t) = -Y(t)A_{12}Y(t) - Y(t)A_{11} + A_{22}Y(t) + A_{21},\ Y(0) = 0. \tag{1·4}$$

Our main result (Theorem 3·4 and Corollary 3·5) is an a-priori estimate for solutions to (1·4):

$$\mathrm{tr}\,Y^*(t)Y(t) \le 4\frac{\mathrm{tr}\,A_{12}A_{21}}{(\mu-\lambda)^2} \tag{1·5}$$

where $\mu, \lambda \in \mathbb{R}$ are such that $A_{11} \leq \lambda \mathbb{1}$ and $\mu \mathbb{1} \leq A_{22}$ with $\lambda < \mu$ or $A_{11} \geq \lambda \mathbb{1}$ and $\mu \mathbb{1} \geq A_{22}$ with $\lambda > \mu$ in the sense of quadratic forms. This enables us to extend the locally defined solution, the existence of which is ensured by the Picard-Lindelöf Theorem, to a globally defined solution. (1·5) may also be used to derive a first, rough bound for the determinant $\det U_{11}(T)$. But its actual importance is the conclusion $\det U_{11}(T) \neq 0$ rather than the quantitative statement.

The intense study of section determinants of linear operators was triggered off by Szegö, who was concerned with the asymptotic behaviour of section determinants of infinite Toeplitz matrices [**8**].

In a comparably abstract framework, section determinants of exponential-like operators were first investigated by Widom [**9**] who considered operators e^X with X being trace class. His main tool was a factorization theorem based upon a general addition formula for operator-valued exponential functions known as Baker-Campbell-Hausdorff formula.

The RDE in (1·4) was studied earlier by Bart, Gohberg, and Kaashoek [**1**] who developed a solution theory for arbitrary initial conditions and gave explicite formulas for the solution. But they made heavily use of the inverse $U_{11}^{-1}(T)$. Therefore, their approach does not fit our purposes because we are mainly interested in the determinant $\det U_{11}(T)$ and computing the inverse $U_{11}^{-1}(T)$ is as difficult as studying the determinant $\det U_{11}(T)$. Thus, the main problem is not to walk in circles. This means in particular that all statements about the solution $Y(t)$ ought to solely involve the operator A and its components rather than $U(t)$ or any of its components.

In Section 4, we sketch an alternative proof of (1·5) that does not use the RDE but just employs plain linear algebra. At a first sight, this more algebraic method seems to be simpler than that based upon the RDE. But we think that the differential equation techniques may be generalized to give more detailed insight in the behaviour of $\det U_{11}(T)$ whereas the stationary method is exhausted by (1·5). As a first token, we refer to the conjecture (6·1).

2. *Section determinants*

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be complex Hilbert spaces with $\dim \mathcal{H}_1 < \infty$. We consider the orthogonal sum $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$. Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded self-adjoint operator:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \ A_{jk} : \mathcal{H}_k \to \mathcal{H}_j, \ j, k = 1, 2. \tag{2·1}$$

Note, that the self-adjointness of A implies $A_{jk}^* = A_{kj}$, $j, k = 1, 2$, where * denotes the adjoint. We restrict ourselves to bounded operators in order to avoid annoying technicalities that would only distract from the basic ideas. Actually, most of the questions that are discussed here lack answers, even in the case of finite dimensional matrices.

In what follows, we shall often deal with operator-valued functions $t \mapsto F(t)$, $t \in \mathbb{R}$, where $F(t) : \mathcal{K} \to \mathcal{K}'$ and $\mathcal{K}$, $\mathcal{K}'$ are Hilbert spaces. Continuity and differentiability of those functions always refer to the usual operator norm. We shall sometimes drop the variable t for convenience and use F instead of $F(t)$ although this is not quite correct from a rigorous point of view. But this should not lead to any misunderstandings.

The unitary operators we want to study are given to us as solutions to a Schrödinger equation:

$$i\frac{d}{dt}U(t) = AU(t),\ U(0) = \mathbb{1}, \tag{2·2}$$

where $\mathbb{1} : \mathcal{H} \to \mathcal{H}$ is the identity operator. We are looking for operator-valued functions $U(t)$, $t \in \mathbb{R}$, that depend continuously differentiably on t. Whenever speaking of solutions to differential equations throughout this paper, we always mean continuously differentiable functions. It is well-known from the theory of evolution equations (see e.g. [**7**]) that there is a unique solution $U(t)$ to this initial value problem. For every $t \in \mathbb{R}$, $U(t) : \mathcal{H} \to \mathcal{H}$ is a bounded unitary operator.

We write $U(t)$ as block matrix:

$$U(t) = \begin{pmatrix} U_{11}(t) & U_{12}(t) \\ U_{21}(t) & U_{22}(t) \end{pmatrix},\ U_{jk}(t) : \mathcal{H}_k \to \mathcal{H}_j,\ j,k = 1,2. \tag{2·3}$$

We need to write the Schrödinger equation (2·2) in components, too:

$$\begin{pmatrix} U'_{11}(t) & U'_{12}(t) \\ U'_{21}(t) & U'_{22}(t) \end{pmatrix} = \begin{pmatrix} A_{11}U_{11}(t) + A_{12}U_{21}(t) & A_{11}U_{12}(t) + A_{12}U_{22}(t) \\ A_{21}U_{11}(t) + A_{22}U_{21}(t) & A_{21}U_{12}(t) + A_{22}U_{22}(t) \end{pmatrix},$$
$$\begin{pmatrix} U_{11}(0) & U_{12}(0) \\ U_{21}(0) & U_{22}(0) \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \tag{2·4}$$

Here a prime denotes the derivative with respect to t. We are interested in the determinant $\det U_{11}(T)$. Our investigations are based upon a simple formula for the logarithmic derivative of the determinant, which can be found in [**3**]. Henceforth, we shall assume $T \geq 0$ although this is not always necessary. But it helps to keep the presentation as simple as possible.

Lemma 2·1. *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded, self-adjoint operator as in* (2·1) *and let $U(t)$ be the solution to the initial value problem* (2·2). *Then the initial value problem for the RDE:*

$$iY'(t) = -Y(t)A_{12}Y(t) - Y(t)A_{11} + A_{22}Y(t) + A_{21},\ Y(0) = 0,\ 0 \leq t \leq T, \tag{2·5}$$

admits exactly one solution $Y(t) : \mathcal{H}_1 \to \mathcal{H}_2$, which for each fixed t is a bounded operator, if and only if $\det U_{11}(t) \neq 0$ for $0 \leq t \leq T$ in which case this section determinant can be expressed through the following formula:

$$\det U_{11}(T) = e^{-iT\,\mathrm{tr}\,A_{11}} \exp\left[-i\int_0^T \mathrm{tr}\,A_{12}Y(t)\,dt\right] \tag{2·6}$$

Proof. The equivalence of the unique solubility of (2·5) and the determinant not vanishing follows immediately from [**1**], Lemma 8.1. Assume now $\det U_{11}(t) \neq 0$, $0 \leq t \leq T$. Since $U_{11}(t)$ is finite dimensional we can use the general formula for the logarithmic derivative of determinants as given in [**3**], Page 163, Item 9:

$$\frac{d}{dt}\ln\det U_{11}(t) = \mathrm{tr}\,U'_{11}(t)U_{11}^{-1}(t).$$

This can be rewritten in the follwoing way:

$$\begin{aligned}\det U_{11}(T) &= \exp\left[\int_0^T \operatorname{tr}\left(U'_{11}(t)U_{11}^{-1}(t)\right)\, dt\right] \\ &= \exp\left[-i\int_0^T \operatorname{tr}\left((A_{11}U_{11}(t)+A_{12}U_{21}(t))U_{11}^{-1}(t)\right)\, dt\right] \\ &= e^{-iT\operatorname{tr}A_{11}}\exp\left[-i\int_0^T \operatorname{tr}\left(A_{12}U_{21}(t)U_{11}^{-1}(t)\right)\, dt\right]\end{aligned}$$

where we used (2·4). We define $Y(t) := U_{21}(t)U_{11}^{-1}(t)$ and recall once again [**1**], Lemma 8.1, to see that $Y(t)$ satisfies (2·5). This completes the proof. □

Note that the initial condition $U_{11}(0) = \mathbb{1}$ and the continuity of $U_{11}(t)$ imply the existence of some $T > 0$ such that $\det U_{11}(t) \neq 0$ for $0 \leq t \leq T$. Thus, the statement of the lemma is not empty. If the RDE possesses a global solution then the determinant does not vanish for all T which is an immediate consequence from Lemma 2·1 above.

LEMMA 2·2. *If $Y(t)$, $t \geq 0$, is a global solution to the initial value problem* (2·5) *for the RDE then* $\det U_{11}(T) \neq 0$ *for all $T \geq 0$ and Lemma* 2·1 *holds for all $T \geq 0$.*

Proof. See Lemma 2·1. □

The Picard-Lindelöf Theorem (see e.g. [**4**]. The proof given therein can be generalized to operator-valued functions in an obvious way.) ensures existence and uniqueness of a locally defined solution to (2·5). The existence of a global solution can be established by providing an a-priori estimate. In order to formulate it properly, we introduce a special kind of norm for operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Let $S_1, S_2 : \mathcal{H}_1 \to \mathcal{H}_2$ be bounded linear operators. It is easily checked that $\operatorname{tr} S_1^* S_2$ has all the properties of a scalar product. The induced norm is equivalent to the usual operator norm since $\mathcal{H}_1$ is finite dimensional. Now by an a-priori estimate we mean that there is a constant $C \geq 0$ such that whenever there is a solution $Y(t)$ to (2·5) on an arbitrary intervall $[0,T]$ it automatically satisfies $\operatorname{tr} Y(t)^* Y(t) \leq C$ on that intervall.

LEMMA 2·3. *Let $Y(t)$ solve* (2·5) *on* $[0,T]$ *and let* $C \geq 0$. *If there is an a-priori estimate*

$$\operatorname{tr} Y^*(t)Y(t) \leq C, \;\; 0 \leq t \leq T \tag{2·7}$$

in the sense explained above then $Y(t)$ can be extended to all $t \geq 0$ yielding the unique solution to (2·5).

Proof. See [**1**], Corollary 8.2. □

Our next task will be providing such an a-priori estimate which will be done in the next section.

3. *Solvability criterion*

We derive an a-priori estimate for solutions to the initial value problem (2·5) in the sense of Section 2. Let $Y(t)$ be a solution to (2·5) on the interval $[0,T]$, $T > 0$. Furthermore, let $S : \mathcal{H} \to \mathcal{H}$ be a bounded operator with components $S_{jk} : \mathcal{H}_k \to \mathcal{H}_j$, $j,k = 1,2$.

We define the operator $W(t) : \mathcal{H}_1 \to \mathcal{H}_1$ by:

$$W := Y^* S_{22} Y + Y^* S_{21} + S_{12} Y + S_{11}. \tag{3·1}$$

By choosing S appropriately, we can make W satisfy a linear differential equation.

LEMMA 3·1. *Let $Y(t)$ solve the initial value problem (2·5) on $[0, T]$, $T > 0$, and let $AS = SA$. Then $W(t)$ solves the initial value problem*

$$iW'(t) = (Y^*(t)A_{21} + A_{11})W(t) - W(t)(A_{12}Y(t) + A_{11}),\ W(0) = S_{11}, \tag{3·2}$$

for $0 \le t \le T$.

Proof. The initial condition follows immediately from $Y(0) = 0$. $W(t)$ depends continuously differentiably on t since $Y(t)$ does. We check the differential equation:

$$\begin{aligned}
iW' &= i(Y^*)' S_{22} Y + iY^* S_{22} Y' + i(Y^*)' S_{21} + iS_{12} Y' \\
&= -(-Y^* A_{21} Y^* - A_{11} Y^* + Y^* A_{22} + A_{12}) S_{22} Y \\
&\quad + Y^* S_{22}(-Y A_{12} Y - Y A_{11} + A_{22} Y + A_{21}) \\
&\quad - (-Y^* A_{21} Y^* - A_{11} Y^* + Y^* A_{22} + A_{12}) S_{21} + S_{12}(-Y A_{12} Y - Y A_{11} + A_{22} Y + A_{21}) \\
&= (Y^* A_{21} + A_{11})(Y^* S_{22} Y + Y^* S_{21}) - (Y^* S_{22} Y + S_{12} Y)(A_{12} Y + A_{11}) \\
&\quad + Y^*(S_{22} A_{22} - A_{22} S_{22}) Y + (S_{12} A_{22} - A_{12} S_{22}) Y + Y^*(S_{22} A_{21} - A_{22} S_{21}) \\
&\quad + S_{12} A_{21} - A_{12} S_{21} \\
&= (Y^* A_{21} + A_{11})(Y^* S_{22} Y + Y^* S_{21} + S_{12} Y + S_{11}) \\
&\quad - (Y^* S_{22} Y + S_{12} Y + Y^* S_{21} + S_{11})(A_{12} Y + A_{11}) \\
&\quad + Y^*(S_{22} A_{22} - A_{22} S_{22} + S_{21} A_{12} - A_{21} S_{12}) Y \\
&\quad + (S_{12} A_{22} - A_{12} S_{22} + S_{11} A_{12} - A_{11} S_{12}) Y + Y^*(S_{22} A_{21} - A_{22} S_{21} + S_{21} A_{11} - A_{21} S_{11}) \\
&\quad + S_{12} A_{21} - A_{12} S_{21} + S_{11} A_{11} - A_{11} S_{11}.
\end{aligned}$$

In order that this becomes a differential equation for W, it is sufficient to require that the following equations are satisfied:

$$\begin{aligned}
S_{22} A_{22} - A_{22} S_{22} + S_{21} A_{12} - A_{21} S_{12} &= 0, \\
S_{12} A_{22} - A_{12} S_{22} + S_{11} A_{12} - A_{11} S_{12} &= 0, \\
S_{22} A_{21} - A_{22} S_{21} + S_{21} A_{11} - A_{21} S_{11} &= 0, \\
S_{12} A_{21} - A_{12} S_{21} + S_{11} A_{11} - A_{11} S_{11} &= 0.
\end{aligned}$$

But this is easily seen to be equivalent to $SA - AS = 0$ whence the lemma follows. □

The special form of Equation (3·2) is due to the quadratic character of W. Therefore it is not surprising that W can be written as product of solutions to an ordinary Schrödinger equation.

LEMMA 3·2. *Let $V(t) : \mathcal{H}_1 \to \mathcal{H}_1$ be the solution to the initial value problem:*

$$iV'(t) = (Y^*(t)A_{21} + A_{11})V(t),\ V(0) = \mathbb{1}, \tag{3·3}$$

for $0 \le t \le T$. Then for $W(t)$ from Lemma 3·1 we have $W(t) = V(t)S_{11}V^*(t)$ for $0 \le t \le T$.

Proof. We verify that $V(t)S_{11}V^*(t)$ solves the same initial value problem (3·2) as $W(t)$. Since (3·2) can possess at most one solution, the statement follows.

The initial condition is obvious. Showing that $VS_{11}V^*$ satisfies the differential equation is merely a simple exercise in differentiating:

$$i\frac{d}{dt}(VS_{11}V^*) = iV'S_{11}V^* + iVS_{11}(V^*)' = (Y^*A_{21}+A_{11})VS_{11}V^* - VS_{11}V^*(A_{12}Y+A_{11}).$$

This completes the proof. $\square$

Note that V can be identified with U_{11}^{-1}. But once again we emphasize that it is our aim not to use any component of U explicitely.

In order to apply the above Lemma 3·2, we need some S having the required property. There are some natural choices for S of which the most simple ones are $S = \mathbb{1}$ and $S = A$ yielding:

$$W_0 := Y^*Y + \mathbb{1}, \; W_1 := Y^*A_{22}Y + A_{12}Y + Y^*A_{21} + A_{11}. \tag{3·4}$$

Actually, this will already serve for our purposes. Note, that W_0 and W_1 are self-adjoint.

The analysis part is now finished. From now on linear algebra will do the job. First of all, we need some lemmas. In the sequel, the relation $B \leq C$ between two self-adjoint operators B and C is always to be understood in the sense of quadratic forms.

LEMMA 3·3. *Let $B : \mathcal{H}_1 \to \mathcal{H}_j$ and $C : \mathcal{H}_j \to \mathcal{H}_j$ be bounded linear operators with either $j = 1$ or $j = 2$. We assume that C is self-adjoint with $C \geq \gamma\mathbb{1}$. Then,*

$$\operatorname{tr} B^*CB \geq \gamma \operatorname{tr} B^*B. \tag{3·5}$$

Proof. In order to compute the trace, we choose a complete orthonormal system $\{\varphi_1, \ldots, \varphi_N\}$ in $\mathcal{H}_1$. Then

$$\operatorname{tr} B^*CB = \sum_{j=1}^{N}(\varphi_j, B^*CB\varphi_j) = \sum_{j=1}^{N}(B\varphi_j, CB\varphi_j) \geq \gamma\sum_{j=1}^{N}(B\varphi_j, B\varphi_j) = \gamma \operatorname{tr} B^*B.$$

The lemma is proved. $\square$

Now we are equipped to prove the main result.

THEOREM 3·4. *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded self-adjoint operator as in* (2·1)*. Let $A_{11} \leq 0$ and $A_{22} \geq \mu\mathbb{1}$ with $\mu > 0$. Then there is an a-priori estimate*

$$\operatorname{tr} Y^*(t)Y(t) \leq 4\frac{\operatorname{tr} A_{12}A_{21}}{\mu^2} \tag{3·6}$$

for every solution $Y(t)$ to the initial value problem (2·5)*. In particular,* (2·5) *possesses a globally defined unique solution.*

Proof. Let W_0 and W_1 be defined as in (3·4). From $W_0 = VV^*$ we conclude:

$$VV^* = \mathbb{1} + Y^*Y.$$

This implies that $VV^* \geq \mathbb{1}$, and that V and V^* are injective. From [**2**], page 38, we infer $V^*V \geq \mathbb{1}$. With the aid of Lemma 3·3, we can estimate since $A_{11} \leq 0$:

$$-\operatorname{tr} W_1 = -\operatorname{tr} VA_{11}V^* = \operatorname{tr}(-A_{11})^{1/2}V^*V(-A_{11})^{1/2} \geq -\operatorname{tr} A_{11}.$$

We recall the definition of W_1:

$$\operatorname{tr} A_{11} \geq \operatorname{tr} W_1 = \operatorname{tr}(Y^*A_{22}Y + A_{12}Y + Y^*A_{21} + A_{11})$$

and conclude:

$$\operatorname{tr} Y^* A_{22} Y \leq -\operatorname{tr}(A_{12}Y + Y^* A_{21}).$$

Now apply Lemma 3·3:

$$\operatorname{tr} Y^* A_{22} Y \geq \mu \operatorname{tr} Y^* Y$$

to obtain:

$$\mu \operatorname{tr} Y^* Y \leq 2(\operatorname{tr} Y^* Y \operatorname{tr} A_{12} A_{21})^{1/2}$$

where we have used the Cauchy-Schwarz inequality. If $\operatorname{tr} Y^* Y \neq 0$, we may divide:

$$\operatorname{tr} Y^* Y \leq 4 \frac{\operatorname{tr} A_{12} A_{21}}{\mu^2}$$

which is obviously correct for $\operatorname{tr} Y^* Y = 0$, too. This proves (3·6). Lemma 2·3 yields the existence of a globally defined solution. □

It is a simple task to extend Theorem 3·4 to hold under more general gap conditions.

COROLLARY 3·5. *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded self-adjoint operator as in* (2·1). *Let either $A_{11} \leq \lambda \mathbb{1}$ and $\mu \mathbb{1} \leq A_{22}$ with $\lambda < \mu$ or $A_{11} \geq \lambda \mathbb{1}$ and $\mu \mathbb{1} \geq A_{22}$ with $\lambda > \mu$. Then there is an a-priori estimate*

$$\operatorname{tr} Y^*(t) Y(t) \leq 4 \frac{\operatorname{tr} A_{12} A_{21}}{(\mu - \lambda)^2} \tag{3·7}$$

for every solution $Y(t)$ to the initial value problem (2·5). *In particular,* (2·5) *possesses a globally defined unique solution.*

Proof. We begin with the case $\lambda < \mu$. We have:

$$\begin{aligned} iY' &= -Y A_{12} Y - Y A_{11} + A_{22} Y + A_{21} \\ &= -Y A_{12} Y - Y(A_{11} - \lambda \mathbb{1}) + Y(A_{22} - \lambda \mathbb{1}) + A_{21}. \end{aligned}$$

The operators $\tilde{A}_{11} := A_{11} - \lambda \mathbb{1}$ and $\tilde{A}_{22} := A_{22} - \lambda \mathbb{1}$ satisfy the assumptions of Theorem 3·4 which yields the statement in this case.

In order to treat the case $\lambda > \mu$, apply the first part to $-A$. This completes the proof. □

We use Corollary 3·5 to derive a bound for $\det U_{11}(T)$.

COROLLARY 3·6. *Under the assumptions of Corollary* 3·5, *we have:*

$$\exp\left[-2T \frac{\operatorname{tr} A_{12} A_{21}}{|\mu - \lambda|}\right] \leq |\det U_{11}(T)| \leq \exp\left[2T \frac{\operatorname{tr} A_{12} A_{21}}{|\mu - \lambda|}\right] \tag{3·8}$$

for $T \geq 0$.

Proof. Follows immediately from $|\operatorname{tr} A_{12} Y|^2 \leq \operatorname{tr}(A_{12} A_{21}) \operatorname{tr}(Y^* Y)$. □

Global solvability of RDE has also been studied in the framework of control theory. However, while we are mainly motivated by problems arising from section determinants there minimizing certain functionals is in the foreground which leads to rather different conditions to be imposed on the equations considered. We want to exemplarily illustrate this by briefly discussing the result from [**5**] on global existence. To be as close as possible to the notation used in [**5**] we write the matrix A in (2·1) as

$$A = \begin{pmatrix} -\hat{A} & \hat{B} \\ \hat{Q} & \hat{A}^* \end{pmatrix}.$$

We assume that $\hat{A}$, $\hat{B}$, and $\hat{Q}$ are $N \times N$ matrices with $\hat{B}$ and $\hat{Q}$ being self-adjoint. If in addition $\hat{B}, \hat{Q} \geq 0$ then it is proved in [**5**], Theorem 16.4.3, that the corresponding RDE (2·5) admits a global solution. It is clear at first sight that this result requires a completely different structure of the matrix A than that taken by us. In particular, the operator $U(t)$ given by (2·2) is no longer unitary which was the main source of our proofs.

4. *An alternative method using linear algebra*

The solution $Y(t)$ to the initial value problem (2·5) was originally defined as $Y(t) = U_{21}(t)U_{11}^{-1}(t)$. We will sketch a method for deriving the bound on $\operatorname{tr} Y^*(t)Y(t)$ in Theorem 3·4 that does not make use of the RDE.

We consider a slightly more general situation. Let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator and let $A : \mathcal{H} \to \mathcal{H}$ be a bounded self-adjoint operator that commutes with U, i.e., $AU = UA$. As in Section 2, we write U and A as block matrices corresponding to the orthogonal sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.

We assume that U_{11}^{-1} exists whence we may define $Y := U_{21}U_{11}^{-1}$. The relations $\mathbb{1} = U^*U$ and $A = U^*AU$ follow immediately from the assumptions. The respective left upper components are

$$\mathbb{1} = U_{11}^*U_{11} + U_{21}^*U_{12} = U_{11}^*(\mathbb{1} + (U_{21}U_{11}^{-1})^*U_{21}U_{11}^{-1})U_{11} \tag{4·1}$$

and

$$\begin{aligned} A_{11} &= U_{11}^*A_{11}U_{11} + U_{11}^*A_{12}U_{21} + U_{21}^*A_{21}U_{11} + U_{21}^*A_{22}U_{21} \\ &= U_{11}^*(A_{11} + A_{12}U_{21}U_{11}^{-1} + (U_{21}U_{11}^{-1})^*A_{21} + (U_{21}U_{11}^{-1})^*A_{22}U_{21}U_{11}^{-1})U_{11}. \end{aligned} \tag{4·2}$$

From equations (4·1) and (4·2), we derive

$$(U_{11}^{-1})^*U_{11}^{-1} = \mathbb{1} + Y^*Y \tag{4·3}$$

and

$$(U_{11}^{-1})^*A_{11}U_{11}^{-1} = A_{11} + A_{12}Y + Y^*A_{21} + Y^*A_{22}Y. \tag{4·4}$$

The right-hand sides of these equations are exactly the expressions W_0 and W_1 as defined in (3·4). Hence, we have a factorization of W_0 and W_1:

$$(U_{11}^{-1})^*U_{11}^{-1} = W_0 \tag{4·5}$$

$$(U_{11}^{-1})^*A_{11}U_{11}^{-1} = W_1. \tag{4·6}$$

We can now proceed exactly as in the proof of Theorem 3·4 in order to derive the desired estimate for $\operatorname{tr} Y^*Y$.

5. *An example*

In order to illustrate that the spectral condition in Theorem 3·4 or Corollary 3·5 generally cannot be dropped without the result becoming incorrect we study a simple example. We choose the underlying Hilbert space to be $\mathcal{H} := \mathbb{C}^2$ and consider the self-adjoint operator $A : \mathcal{H} \to \mathcal{H}$,

$$A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which obviously does not satisfy the condition of Theorem 3·4 or Corollary 3·5. With this choice the unitary operator U given by (2·2) and particularly the component U_{11} can be computed explicitely. Once more differentiating in (2·2) yields

$$U'' = -A^2 U = -U$$

with initial conditions $U(0) = \mathbb{1}$ and $U'(0) = -iA$. For U_{11} we obtain

$$U_{11}' = -U_{11}, \ U_{11}(0) = 1, \ U_{11}'(0) = 0,$$

whereof the solution is

$$U_{11}(t) = \cos t.$$

Because U_{11} clearly has zeroes Lemma 2·3 tells us that there cannot be a globally defined solution to the RDE. Thus, the spectral condition of Theorem 3·4 should not be too far from being necessary.

6. *Remarks*

We mention some points that might be worth further investigations.

The i in equation (2·5) was crucial as can be seen by considering Lemma 3·1. Otherwise, we would have arrived at the condition $AS + SA = 0$ instead of $AS - SA = 0$. Generally, this anti-commutator equation will not admit a non-trivial solution as the example $A = \mathbb{1}$ shows. Thus, in order to handle other equations, additional techniques are necessary.

The proof of the a-priori estimate in Theorem 3·4 as well as the algebraic method in Section 4 were based upon considering the expressions W_0 and W_1 which involved powers of A up to order 1. Of course, one could think about more general criteria by employing higher powers A^n or even more complicated expressions. This will work as long as the commutation condition in Theorem 3·4 is satisfied.

In the proof of Theorem 3·4, we started by estimating $\operatorname{tr} W_1$. The relations

$$\operatorname{tr} W_1 = \operatorname{tr} V A_{11} V^*, \ \operatorname{tr} A_{11} W_0 = \operatorname{tr} A_{11} V V^* = \operatorname{tr} V^* A_{11} V$$

suggest that there is a more exact relationship. Indeed, computing the first Taylor coefficients leads to the conjecture

$$\operatorname{tr} W_1 = \operatorname{tr} A_{11} W_0. \tag{6·1}$$

We think that proving this equality requires more algebraic techniques.

The bounds for the determinant in Corollary 3·6 are very rough and far from optimal because the upper bound becomes larger as $T \to +\infty$. This is in contrast to the inequality $|\det U_{11}(T)| \leq 1$, which can be derived from the unitarity of $U(T)$. Analogously the lower bound tends to zero. Thus the bounds will not reflect the exact asymptotic behaviour at all. The reason is that the integral $\int_0^T \operatorname{tr} A_{12} Y(t)\, dt$ is bounded because of the oscillations of the integrand which was completely neglected since the estimate was obtained using the absolute value $Y(t)$. Thus, in order to derive more precise bounds, we need information about the sign of $\operatorname{tr} A_{12} Y(t)$. A first step towards more powerful methods should be the proof of the conjectured equality (6·1).

Finally, we want to point out that there are other ways of describing the determinant $\det U_{11}(T)$. For instance, the exponent in (2·6) can be characterized through a linear integral equation rather than a RDE, see [**6**].

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