Diagonalization of the Hilbert matrix

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Hilbert matrix

$$H = \left(\frac{1}{j+k}\right)_{j,k\in\mathbb{N}}$$

bounded self-adjoint operator on $l^2(\mathbb{N})$.

Formal eigenvectors

Find $(c_1, c_2, \ldots)$ ($\notin l^2(\mathbb{N})$ possibly) with

$$\sum_{k=1}^{\infty} \frac{1}{j+k} c_k = \mu c_j, \ j \in \mathbb{N},$$

$\mu \in \mathbb{R}$ and the series converges.

Kato: existence of formal eigenvectors.

Rosenblum: formal eigenvectors via Laguerre cofficients of Whittaker functions. Main tool: write H as integral operator.

Alternative idea

Formal eigenvectors satisfy a second order difference equation.

Step 1: Commuting operator

The bounded self-adjoint operator

$$G : l^2(\mathbb{N}) \to l^2(N),\ G_{jk} := \frac{1}{\max\{j,k\}}$$

commutes with H, $HG = GH$.

Step 2: Formal eigenvectors of G

Eigenvalue equation (normalized by $c_1 = 1$)

$$\frac{1}{\lambda} c_j = \sum_{k=1}^{\infty} \frac{1}{\max\{j,k\}} c_k = \frac{1}{j} \sum_{k=1}^{j} c_k + \sum_{k=j+1}^{\infty} \frac{1}{k} c_k$$

can be rewritten

$$(j+1)jc_{j+1} - 2j^2 c_j + j(j-1)c_{j-1} = \lambda c_j. \quad (D)$$

Initial conditions $c_0 = 0$, $c_1 = 1$. Always solvable. Candidates for formal eigenvectors of H.

Remark:

Difference equation for continuous dual Hahn polynomials (special case of Wilson's polynomials).

Step 3: Asymptotics

Asymptotics of c_j in (D) following Wong et.al.

Characteristic equation

$$\gamma^2 - \gamma - \lambda = 0, \ \gamma_{1,2} = \frac{1}{2}(1 \pm \sqrt{1+4\lambda}).$$

1. $\lambda \neq -\frac{1}{4}$: $c_j \sim A_1 j^{\gamma_1 - 1} + A_2 j^{\gamma_2 - 1}$,
   (a) $\lambda < -\frac{1}{4}$: $\operatorname{Re}\gamma_{1,2} = \frac{1}{2}$.
   (b) $\lambda > -\frac{1}{4}$: $\gamma_{1,2} \in \mathbb{R}$

2. $\lambda = -\frac{1}{4}$: $c_j \sim A\frac{\ln j}{\sqrt{j}}$, $A \neq 0$

Theorem 1. *For $\lambda < 0$ the corresponding c_j are formal eigenvectors of H.*

Proof. Calculations in reversed order. Note the asymptotics. $\square$

Theorem 2. *The operator* $D : l_0^2(\mathbb{N}) \to l^2(\mathbb{N})$

$$(D\varphi)_j = (j+1)j\varphi_{j+1} - 2j^2\varphi_j + j(j-1)\varphi_{j-1}$$

- *is essentially self-adjoint*
- *has spectrum* $\sigma(D) =]-\infty, -\frac{1}{4}]$
- *commutes (formally) with* H*.*

Proof. We have:

- Self-adjointness: Wouk's criterion.
- Spectrum: Build Weyl sequences from c_j and use the asymptotics.
- Commutation property: straighforward.

$\square$

Spectral map τ

Determine $\mu = \tau(\lambda)$ such that

$$Dc = \lambda c \longleftrightarrow Hc = \mu c$$

Generating function

For a solution to (D)

$$u(x) := \sum_{j=0}^{\infty} c_j x^j$$

is well-defined and satisfies

$$x(1-x)[(1-x)u'' - 2u'] = \lambda u$$

and

$$u(0) = 0,\ u'(0) = 1.$$

Note $c_0 = 0,\ c_1 = 1$.

Solution is a hypergeometric function

$$u(x) = (1-x)^{\alpha} x F(\alpha+1, \alpha+2; 2; x),$$
$$\alpha = -\frac{1}{2}(1+\sqrt{1+4\lambda})$$

Theorem 3. *The spectral map* τ *is given by*

$$\tau : \sigma(D) \to \sigma(H), \ \tau(\lambda) = \frac{\pi}{\cosh \pi\sqrt{-\frac{1}{4} - \lambda}}.$$

In particular $\tau(-\frac{1}{4}) = \pi$ *and* $\tau(-\infty) = 0$.

Proof. Note $c_0 = 0$, $c_1 = 1$. Then,

$$\begin{aligned}
\tau(\lambda) &= (Hc)_1 \\
&= \sum_{j=0}^{\infty} \frac{1}{j+1} c_j \\
&= \int_0^1 u(x)\, dx \\
&= \int_0^1 (1-x)^\alpha x F(\alpha+1, \alpha+2; 2; x)\, dx
\end{aligned}$$

To evaluate the integral use

$$\begin{aligned}
F(\alpha+1, \alpha+2; 2; x) = {} & \frac{1}{B(1+\alpha, 1-\alpha)} \times \\
& \times \int_0^1 t^\alpha (1-t)^{-\alpha} (1-xt)^{-(2+\alpha)}\, dt.
\end{aligned}$$

$\square$

Application: Hilbert's inequality

$$\sum_{j,k=1}^{\infty} \frac{1}{j+k} \bar{\varphi}_j \varphi_k \le \pi \sum_{j=1}^{\infty} |\varphi_j|^2.$$

Equivalent to saying $\rho := \sup \sigma(H) = \pi$.

Problem: ρ_n for $n \times n$-Hilbert matrices H_n.

Idea: Let $c_j^{(n)}$ be the eigenvector of H_n to ρ_n. Note $c_j^{(n)} \ge 0$ by Perron-Frobenius. Define

$$u(x) = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{n} c_j^{(n)} e^{ijx},$$

$$a(x) = i(x - \operatorname{sign}(x)) = \sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{1}{j} e^{-ijx}$$

Note $\|u\| = \|c^{(n)}\|$ and $\|a\|_\infty = \pi$. Then,

$$\begin{aligned} \rho_n \|c^{(n)}\|^2 &= \sum_{j,k=1}^{n} \frac{1}{j+k} \bar{c}_j^{(n)} c_k^{(n)} \\ &= \int_{-\pi}^{\pi} a(x) \bar{u}(-x) u(x)\, dx. \end{aligned}$$

Estimate the integral by Cauchy-Schwarz.

Refined Cauchy-Schwarz inequality:

$$|(f,g)|^2 \leq [\|f\|\|g\|]^2 - \|(h,g)f - (h,f)g\|^2,$$

with $\|h\| = 1$. Choose $h \equiv \frac{1}{\sqrt{2\pi}}$. Then,

$$\begin{aligned}
\rho_n^2 \|c^{(n)}\|^4 &\leq \|a\|_\infty^2 \|u\|^4 - \left[\sum_{k=1}^{n} \frac{1}{k} c_k^{(n)}\right]^2 \|u\|^2 \\
&\leq \pi^2 \|u\|^4 - \left[\sum_{k=1}^{n} \frac{1}{k+1} c_k^{(n)}\right]^2 \|u\|^2 \\
&= \pi^2 \|u\|^4 - \rho_n^2 [c_1^{(n)}]^2 \|u\|^2
\end{aligned}$$

Kato's estimate:

$$0 \leq c_j^{(n)} \leq c_j \leq A \frac{\ln j}{\sqrt{j}}.$$

Improved Hilbert inequality:

$$\rho_n \leq \pi - \frac{B}{\ln^3 n}.$$

De Bruijn-Wilf asymptotics:

$$\rho_n \sim \pi - \frac{\pi^5}{2 \ln^2 n}$$