Liouville-type formulae for section determinants and their relation to Szegő limit theorems.

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Liouville formula

Starting point:

$$\frac{d}{dt} \ln \det S(t) = \operatorname{tr} S'(t) S(t)^{-1} \tag{1}$$

$S(t) \in \mathbb{C}^{n\times n}$ differentiable and non-singular.

Initial value problem:

$$U' = AU, \; U(0) = \mathbb{1},$$

where $A, U(t) \in \mathbb{C}^{n\times n}$.

(1) implies immediately Liouville's formula:

$$\det U(T) = e^{T \operatorname{tr} A}$$

Liouville-type formulae:
Section determinants of $U(T)$.

Szegő limit theorems:
Limit behaviour of determinants of large sections.

Hilbert space: $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.

Block matrices:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \; U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

Section determinant:

$$\det U_{11}(T)$$

Main tool:

$$\frac{d}{dt} \ln \det U_{11} = \operatorname{tr} A_{11} + \operatorname{tr} A_{12} Y \qquad (2)$$

with

$$Y := U_{21} U_{11}^{-1}$$

Follows from (1) and $U' = AU$.

Problems:
1) Does U_{11}^{-1} exist?
2) Find estimates for $\operatorname{tr} A_{12} Y(t)$

Case 1 A symmetric, i.e. $U(t)$ symmetric
Case 2 A anti-symmetric, i.e. $U(t)$ unitary

Cases 1,2 need different representations of Y.

Case 1: Symmetric A

Use variation of constants to obtain

$$U_{21}(t) = \int_0^t e^{(t-\tau)A_{22}} A_{21} U_{11}(\tau)\, d\tau,$$

and eliminate U_{21} from $Y = U_{21}U_{11}^{-1}$. Then,

$$\boxed{Y(t) = \int_0^t e^{(t-\tau)A_{22}} A_{21} U_{11}(\tau) U_{11}(t)^{-1}\, d\tau}$$

Trace estimate:

1) $A_{11}^* = A_{11},\ A_{22}^* = A_{22},\ A_{21}^* = A_{12}$
2) $U(t)$ is symmetric.
3) in $Y(t)$ only $U_{11}(t)^{-1}$ is dangerous.
4) A is bounded. Hence $A \geq \gamma \mathbb{1}$, $\gamma > -\infty$.
5) $U(t) = e^{tA} \geq e^{\gamma t}\mathbb{1}$ and $U_{11}(t) \geq e^{\gamma t}\mathbb{1}$.

Remark: To bound $\|U_{11}(t)^{-1}\|$ independently of $\dim \mathcal{H}_1$ requires A being bounded.

Open question: May A be semi-bounded?

Theorem 1. *For symmetric A:*

$$|\operatorname{tr} A_{12} Y(t)| \le C(t) \operatorname{tr} A_{12} A_{21}$$

where $C(t)$ does not depend on $\dim \mathcal{H}_1$*.*

Szegő limit theorem: Recall that from (2)

$$\left[e^{-T \operatorname{tr} A_{11}} \det U_{11}(T)\right]^{1/n} = \exp\left[\frac{1}{n}\int_0^T \operatorname{tr} A_{12} Y(t)\, dt\right]$$

Take an increasing sequence of subspaces $\mathcal{H}_1^{(n)}$ with $\dim \mathcal{H}_1^{(n)} = n$, $n \in \mathbb{N}$.

Følner condition: $\operatorname{tr} A_{12} A_{21} = o(n)$.

By Theorem 1

$$\lim_{n\to\infty} \left[e^{-T \operatorname{tr} A_{11}} \det U_{11}(T)\right]^{1/n} = 1$$

Følner class: Contains compact operators and Toeplitz operators but not all bounded operators.

Case 2: Anti-symmetric A

Bad news: Previous method fails.
Example:

$$A := -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then

$$U_{11}(t) = \cos t$$

has zeros and $Y(t) = U_{21}(t)U_{11}(t)^{-1}$ does not exist for all t. Hence no estimate for $\operatorname{tr} A_{12}Y(t)$.

Different representation: Check that $Y = U_{21}U_{11}^{-1}$ satisfies the Riccati equation

$$\boxed{Y' = -YA_{12}Y - YA_{11} + A_{22}Y + A_{21}}$$

$$\boxed{Y(0) = 0}$$

Trace estimate (first step): A-priori estimate for Y.

Theorem 2. *Let $A = -iB$ be anti-symmetric. Then,*

$$\operatorname{tr} Y(t)^* Y(t) \leq 4 \frac{\operatorname{tr} B_{12} B_{21}}{|\lambda - \mu|^2}$$

where

$$\lambda := \sup \sigma(B_{11}),\ \mu := \inf \sigma(B_{22}),\ \lambda < \mu$$

or the other way round.

Proof. Consider 'quadratic forms' in Y:

$$\begin{aligned} W_0 &:= Y^* Y + \mathbb{1} \\ W_1 &:= Y^* B_{22} Y + B_{12} Y + Y^* B_{21} + B_{11} \end{aligned}$$

Show that W_0 and W_1 solve linear differential equations and conclude

$$W_0 = V V^*,\ W_1 = V B_{11} V^*$$

where

$$i V' = (Y^* B_{21} + B_{11}) V,\ V(0) = \mathbb{1}$$

When $\lambda = 0$ one can estimate traces elementary starting from the factorisations for W_0 and W_1. For the general case shift the spectrum. $\square$

Trace estimate (second step):

$$|\operatorname{tr} B_{12}Y^*(t)| \leq (\operatorname{tr} B_{12}B_{21})^{1/2}(\operatorname{tr} Y(t)^*Y(t))^{1/2}$$

$$\leq 2\frac{\operatorname{tr} B_{12}B_{21}}{|\lambda - \mu|}$$

Szegő limit theorems:

1) Følner condition alone is not enough.
2) Some relation between Følner and gap condition is needed.
3) The gap condition cannot be satisfied by Toeplitz matrices because A_{22} is somehow the entire matrix A.
4) Application to Schrödinger operators from quantum mechanics (transition probabilities): $A = -i(H_0 + V)$, H_0 has purely discrete spectrum and V is small with respect to the gaps of H_0.