An adiabatic theorem for section determinants of spectral projections

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Fermionic systems

One-particle space $\mathcal{H}$, N-particle space

$$\mathcal{H}_N := \underbrace{\mathcal{H} \wedge \ldots \wedge \mathcal{H}}_{N-\text{times}}$$

Transition probability

$$|(\varphi, \psi)|^2 = |\det((\varphi_j, \psi_k))_{j,k=1,\ldots,N}|^2$$

where

$$\varphi = \varphi_1 \wedge \ldots \wedge \varphi_N,\ \psi = \psi_1 \wedge \ldots \wedge \psi_N \in \mathcal{H}_N$$

Section determinants

Introduce orthogonal projections

$$P_N = \sum_{j=1}^{N} (\varphi_j, \cdot)\varphi_j,\ \Pi_N = \sum_{j=1}^{N} (\psi_j, \cdot)\psi_j$$

then

$$|(\varphi, \psi)|^2 = \det P_N \Pi_N P_N$$

The 'section determinant' refers to $P_N \Pi_N P_N : \operatorname{ran} P_N \to \operatorname{ran} P_N$.

Classical adiabatic theorem

Family of self-adjoint operators $H(s)$
Corresponding spectral projections $P(s)$

Adiabatic theorem

$$P(s) = \lim_{\eta \to 0} U_\eta(s) P(0) U_\eta^*(s)$$

Schrödinger equation

$$i\eta \frac{d}{ds} U_\eta(s) = H(s) U_\eta(s), \; U_\eta(0) = \mathbb{1}$$

Spectral gap for $H(s)$ (standard proof)

We want to study section determinants

$$\det P(0)P(1)P(0)$$

Two possibilities

1) Apply the adiabatic theorem

2) Prove a theorem tailored for section determinants

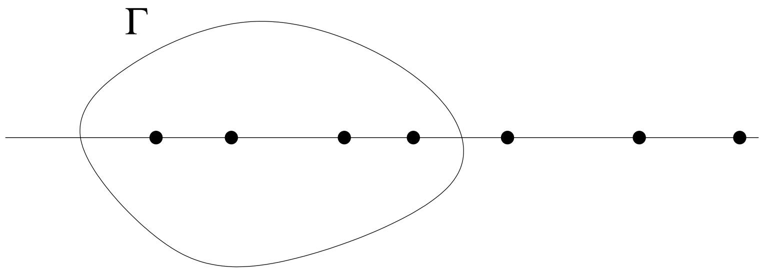
Special adiabatic theorem

Family of self-adjoint operators

$$A(\alpha) := A_0 + \alpha B, \quad \alpha \in [0,1]$$

Assumptions:

- B bounded (for simplicity)
- Gap condition: spectrum of $A(\alpha)$



Same Γ for all α.

Riesz' formula Spectral projection $P(\alpha)$

$$P(\alpha) = \frac{1}{2\pi i} \int_\Gamma (z\mathbb{1} - A(\alpha))^{-1} \, dz$$

Four steps toward the adiabatic theorem.

First step

Logarithmic derivative
$S(\alpha) \in \mathbb{C}^{N \times N}$ differentiable and non-singular:

$$\frac{d}{d\alpha} \ln \det S(\alpha) = \operatorname{tr} S^{-1}(\alpha) S'(\alpha)$$

Orthogonal projections

$$\frac{d}{d\alpha} \ln \det P(0)P(\alpha)P(0) = \\ 2 \operatorname{Re} \operatorname{tr} P^{+}(\alpha)(\mathbb{1} - P(\alpha))P'(\alpha)P(\alpha)$$

Pseudo inverse on $\operatorname{ran} P(0)$

$$P^{+}(\alpha) := (P(0)P(\alpha)P(0))^{-1}$$

Friedrichs integral
By Riesz' formula:

$$P'(\alpha) = \\ \frac{1}{2\pi i} \int_{\Gamma} (z\mathbb{1} - A(\alpha))^{-1} B (z\mathbb{1} - A(\alpha))^{-1} \, dz$$

Schrödinger equation
Relate the resolvents to solutions of

$$i \frac{d}{dt} U(t, \alpha) = A(\alpha) U(t, \alpha), \quad U(0, \alpha) = \mathbb{1}$$

Second step

Laplace transform

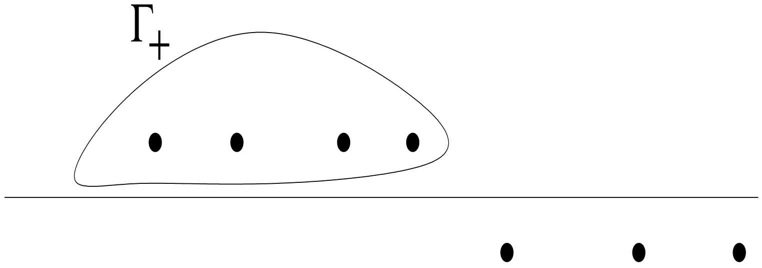
For $\operatorname{Im} w > 0$

$$(w\mathbb{1} - A(\alpha))^{-1} = -i \int_{-\infty}^{0} e^{-iwt} U(-t,\alpha)\, dt$$

Cannot be applied directly.

Trick

Shift the spectrum of $A(\alpha)$ and deform Γ



Shifted resolvents

$((z + i\frac{\eta}{2})\mathbb{1} - A(\alpha))^{-1}$ and $((z - i\frac{\eta}{2})\mathbb{1} - A(\alpha))^{-1}$

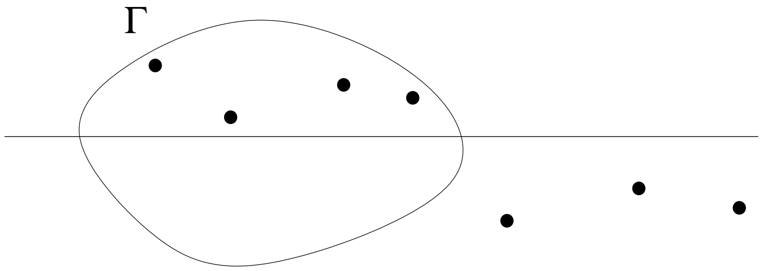
Limit

$$-i \int_{-\infty}^{0} e^{\eta t} U(-t,\alpha)(\mathbb{1} - P(\alpha)) B P(\alpha) U(t,\alpha)\, dt \\ \to (\mathbb{1} - P(\alpha)) P'(\alpha) P(\alpha), \quad \eta \to +0$$

Third step

Problem
Any integral equation would need $U(t,\alpha) \to 0$, $t \to -\infty$. But $\|U(t,\alpha)\| = 1$.

Trick
Shift the spectrum



Shifted operator

$$A_\varepsilon(\alpha) := A(\alpha) + i\varepsilon(2P(0) - \mathbb{1}), \quad \varepsilon > 0$$

Corresponding quantities $U_\varepsilon(t,\alpha)$, $P_\varepsilon(\alpha)$

$$U_\varepsilon(t,\alpha)P_\varepsilon(\alpha) \to 0,\ t \to -\infty.$$

Gap condition
Sufficient dichotomy criterion

$$P(0)A(\alpha)P(0) \le \lambda(\alpha) < \\ < \mu(\alpha) \le (\mathbb{1} - P(0))A(\alpha)(\mathbb{1} - P(0))$$

or the other way round.

Fourth and final step

Idea

After some calculations

$$\int_{-\infty}^{0} e^{\eta t} \operatorname{tr} B U_\varepsilon(t,\alpha) P_\varepsilon(\alpha) P_\varepsilon^{+}(\alpha) U_\varepsilon(-t,\alpha)\, dt$$

Looks like the trace of an integral operator.

Time-ordered Green operator

$G_\varepsilon(t,t';\alpha) =$

$$\begin{array}{lr} t \le t' & iU_\varepsilon(t,\alpha) P_\varepsilon(\alpha) P_\varepsilon^{+}(\alpha) U_\varepsilon(-t',\alpha) \\ t > t' & iU_\varepsilon(t,\alpha)[P_\varepsilon(\alpha) P_\varepsilon^{+}(\alpha) - \mathbb{1}] U_\varepsilon(-t',\alpha) \end{array}$$

Wiener-Hopf equation

$$\begin{aligned} G_\varepsilon(t,t';\alpha) &= e^{-\varepsilon|t-t'|} G_0(t-t') \\ &\quad + \alpha \int_{-\infty}^{0} e^{-\varepsilon|t-\tau|} G_0(t-\tau) B G_\varepsilon(\tau,t';\alpha)\, d\tau \end{aligned}$$

with

$$G_0(t) := \begin{cases} iU(t,0)P(0) & t \le 0 \\ iU(t,0)(P(0) - \mathbb{1}) & t > 0 \end{cases}$$

Proof Use Dyson's equation.

Adiabatic theorem

$$\det(P(0)P(1)P(0)) = \lim_{\eta\to+0}\lim_{\varepsilon\to+0}\exp\Big[-2\,\mathrm{Re}\int_0^1\int_{-\infty}^0 e^{\eta t}\,\mathrm{tr}\,BG_\varepsilon(t,t+0;\alpha)\,dt\,d\alpha\Big]$$

Wiener-Hopf equation

$$G_\varepsilon(t,t';\alpha) = e^{-\varepsilon|t-t'|}G_0(t-t') + \alpha\int_{-\infty}^0 e^{-\varepsilon|t-\tau|}G_0(t-\tau)BG_\varepsilon(\tau,t';\alpha)\,d\tau$$

Free Green operator

$$G_0(t) := \begin{cases} iU(t,0)P(0) & t\le 0 \\ iU(t,0)(P(0)-\mathbb{1}) & t>0 \end{cases}$$

Solution theory

Existence

$$\det P(0)P(\alpha)P(0) \neq 0$$

Uniqueness

Homogeneous equation

$$H_\varepsilon(t;\alpha) = \alpha \int_{-\infty}^{0} e^{-\varepsilon|t-\tau|} G_0(t-\tau) B H_\varepsilon(\tau;\alpha)\, d\tau$$

is equivalent to a boundary value problem

$$i\frac{d}{dt}\varphi(t) = A_\varepsilon(\alpha)\varphi(t)$$
$$P(0)\varphi(0) = 0 \quad \varphi(-\infty) = 0$$

Sufficient criterion

Above gap condition

$$P(0)A(\alpha)P(0) \leq \lambda(\alpha) < \mu(\alpha) \leq (\mathbb{1} - P(0))A(\alpha)(\mathbb{1} - P(0))$$

or the other way round.