An abstract Szegő theorem

Peter Otte

Fakultät für Mathematik, Ruhr-Universität Bochum, Universitätsstraße 150, 44780 Bochum, Germany, Phone +49 (0)234 32 23352, Telefax +49 (0)234 32 14531

Abstract

Given a semi-group $U(t)$ of bounded linear operators with bounded self-adjoint generator A we estimate the logarithm of the section determinants of $U(t)$ in terms of A. When A is subject to an additional condition, which is related to so-called Følner sequences of orthogonal projections, this estimate implies a Szegő type theorem for bounded, self-adjoint, and strictly positive operators. We show that the condition mentioned is satisfied when A is a Toeplitz operator or a compact operator.

1 Introduction

When Szegő [6] published his theorem on the asymptotic behaviour of Toeplitz determinants he initiated an intense research in this area. Many refinements and generalizations have been given since then. Most of the hitherto results rely on some kind of Toeplitz or Toeplitz-like structure. Here we embed the problem into the framework of semi-groups of bounded operators which makes available simple but useful techniques from ordinary differential equations.

To be more precise let $U(t)$, $t \in \mathbb{R}$, be a semi-group of bounded operators acting on a Hilbert space $\mathcal{H}$ with generator A. We assume A to be bounded and self-adjoint. $U(t)$ satisfies the familiar initial value problem

$$U' = AU,\ U(0) = \mathbb{1}. \tag{1}$$

Let $\mathcal{H}$ be written as orthogonal sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and represent $U(t)$ by a 2×2 block matrix:

$$U(t) = \begin{pmatrix} U_{11}(t) & U_{12}(t) \\ U_{21}(t) & U_{22}(t) \end{pmatrix} \tag{2}$$

Email address: `peter.otte@ruhr-uni-bochum.de` (Peter Otte).

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and A analogously. Finally let $N := \dim \mathcal{H}_1$ be finite. Then the section determinant $\det U_{11}(T)$ is well-defined. Starting from the well-known formula for the logarithmic derivative of a determinant and using the differential equation for U one can derive

$$\det U_{11}(T) = e^{T \operatorname{tr} A_{11}} e^{F(T)}. \tag{3}$$

The main outgrowth of our investigations is an estimate of the function F in terms of the generator A. We shall show in Theorem 5

$$|F(T)| \leq C(T) \operatorname{tr} A_{12} A_{21} \tag{4}$$

with some $C(T)$ that contains solely the spectral data of the entire operator A and does not involve the dimension N. In a more general framework Laptev and Safarov [4] proved a related but somewhat different estimate their proof being based upon the Berezin-Lieb inequality. We shall briefly discuss the relationship after Theorem 5.

In order to make (4) the basis for a Szegő theorem we need that $\operatorname{tr} A_{12}A_{21}$ is somehow smaller than $\operatorname{tr} A_{11}$. Thus, let us look at $\operatorname{tr} A_{11}$ first. Since A is bounded so are the diagonal elements of A whence the worst case occurs when all the diagonal entries are the same. This shows

$$\operatorname{tr} A_{11} = O(N) \tag{5}$$

and it is thus reasonable to require

$$\operatorname{tr} A_{12} A_{21} = o(N). \tag{6}$$

Based upon this the abstract Szegő theorem presented in Theorem 6 is an immediate consequence of Formula (3). Hence the task of deriving Szegő theorems is now seen to be reduced to finding a large as possible class of operators such that (6) holds. This class does not comprise all bounded operators as will be illustrated by a simple example in Section 3. However, we shall prove in that section that, at least, compact operators and Toeplitz operators belong to this class thereby recovering the classical first Szegő theorem. Some further types of operators having very specific properties can also be considered. We shall, however, omit treating them because this would not shed any light on the problem of how to determine the general class of operators that satisfy (6). Related problems have been studied in connection with so-called Følner sequences, which we shall touch briefly in Section 3, but not to an extent that is useful for our purposes. The question mentioned thus seems to be an open one.

2 An estimate for the logarithm of section determinants

Let $\mathcal{H}$ be a separable complex Hilbert space. We assume that $\mathcal{H}$ is the orthogonal sum

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \tag{7}$$

of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with $N := \dim \mathcal{H}_1 < \infty$. Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded self-adjoint operator. The boundedness of a A is, first of all, used for the sake of convenience in order to circumvent the technicalities inevitably arising through unbounded operators. The actual reason, however, will become clear from Lemma 2. The operator A can be written as block matrix corresponding to the decomposition (7)

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{8}$$

with $A_{jk} : \mathcal{H}_k \to \mathcal{H}_j$, $j, k = 1, 2$. All block matrices that will appear throughout are to be understood with respect to (7) without further mention. Although the A_{jk} depend on N we shall not indicate this dependence but suppress the index N because it will be treated fixed in this section. The operators A_{jk} are bounded, which follows immediately from the boundedness of A. Let $U(t) : \mathcal{H} \to \mathcal{H}$, $t \in \mathbb{R}$, be the group of bounded linear operators generated by A. Detailed informations on such groups can be found in [5]. Since A is assumed to be bounded $U(t)$ depends continuously differentiably on t with respect to the operator norm. Hereafter analysis of operator-valued functions is always meant with respect to the operator norm. Also thanks to the boundedness of A we can write explicitely $U(t) = \exp(tA)$, $t \in \mathbb{R}$, with the exponential being defined via the usual power series. For the sake of convenience we shall also use the notation e^{tA} instead of $U(t)$. $U(t)$ satisfies the initial value problem

$$U'(t) = AU(t), \; U(0) = \mathbb{1}. \tag{9}$$

The very source of our considerations is the well-known formula for the logarithmic derivative of a determinant. Let $S(t) : \mathcal{H}_1 \to \mathcal{H}_1$ be linear operators that depend continuously differentiably on t and assume $\det S(t) \neq 0$. Then,

$$\frac{d}{dt} \ln \det S(t) = \operatorname{tr} S'(t) S(t)^{-1} \tag{10}$$

respectively

$$\det S(T) = \det S(0) \exp \left[\int_0^T \operatorname{tr} S'(t) S(t)^{-1} \, dt \right], \tag{11}$$

which is valid whenever the inverse exists [2]. In order to apply this to $\det U_{11}(t)$ we first have to show that $U_{11}(t)$ is invertible which is indeed correct due to the fact that the spectrum of an operator acting on the entire Hilbert space $\mathcal{H}$ shrinks when we restrict this operator to the subspace $\mathcal{H}_1$. This observation will be useful at later points, too.

Lemma 1 *Let $S : \mathcal{H} \to \mathcal{H}$ be a bounded self-adjoint operator being written as block matrix in the usual way. Then,*

$$\sup \sigma(S_{jj}) \le \sup \sigma(S), \; \inf \sigma(S_{jj}) \ge \inf \sigma(S), \; j = 1, 2,$$

where σ denotes the spectrum of the respective operators.

PROOF. We have

$$\sup \sigma(S_{11}) = \sup_{\substack{\varphi \in \mathcal{H}_1 \\ \|\varphi\|=1}} (\varphi, S_{11}\varphi) \le \sup_{\substack{\varphi \in \mathcal{H} \\ \|\varphi\|=1}} (\varphi, S\varphi) = \sup \sigma(S).$$

The same holds for the infimum and $j = 2$. $\square$

From the above lemma we deduce immediately that $U_{11}(t)$ can be inverted with the inverse's norm being estimated independently of the dimension N.

Lemma 2 *For each $t \in \mathbb{R}$ the operator $U_{11}(t) : \mathcal{H}_1 \to \mathcal{H}_1$ is invertible and we have*

$$\|U_{11}(t)^{-1}\| \le e^{-\gamma t} \tag{12}$$

with $\gamma := \inf \sigma(A)$.

PROOF. Since A is bounded and self-adjoint we have $\gamma > -\infty$ and by the spectral mapping theorem

$$\inf \sigma(U(t)) \ge e^{\gamma t} > 0$$

which implies with the aid of Lemma 1

$$\inf \sigma(U_{11}(t)) \ge e^{\gamma t}.$$

The Lax-Milgram theorem (see [3]) shows the invertibility of $U_{11}(t)$ and also yields the bound for the inverse. $\square$

The boundedness of A is not necessary to prove that $U(T)$ is invertible. It is enough that A is self-adjoint and generates a semi-group. However, if we dropped the boundedness of A we would not be able to obtain a bound on the inverse that does not depend on the dimension N as show simple examples of diagonal matrices. We can now express our determinant via the above Formula (11)

$$\det U_{11}(T) = \exp\left[\int_0^T \operatorname{tr} U_{11}'(t) U_{11}(t)^{-1}\, dt\right]. \tag{13}$$

In order to cast U_{11}' into a comfortable form we write down the first column of the initial value problem (9)

$$U_{11}' = A_{11}U_{11} + A_{12}U_{21} \tag{14}$$
$$U_{21}' = A_{21}U_{11} + A_{22}U_{21} \tag{15}$$

and

$$U_{11}(0) = \mathbb{1},\ U_{21}(0) = 0. \tag{16}$$

Since we are interested in U_{11}' we kick off U_{21} by regarding (15) as an inhomogeneous initial value problem and solving for U_{21} by the familiar techniques from ordinary differential equations, which also hold in the operator-valued case

$$U_{21}(t) = \int_0^t e^{(t-\tau)A_{22}} A_{21} U_{11}(\tau)\, d\tau. \tag{17}$$

We insert this into (14)

$$U_{11}'(t) = A_{11}U_{11}(t) + \int_0^t A_{12} e^{(t-\tau)A_{22}} A_{21} U_{11}(\tau)\, d\tau. \tag{18}$$

Eventually, (13) becomes

$$\det U_{11}(T) = e^{T \operatorname{tr} A_{11}} \exp\left[\int_0^T \int_0^t \operatorname{tr} A_{12} e^{(t-\tau)A_{22}} A_{21} U_{11}(\tau) U_{11}(t)^{-1}\, d\tau\, dt\right]. \tag{19}$$

This generalizes the well-known Liouville formula from ordinary differential equations. It shows that the off-diagonal elements A_{12} and A_{21} of the generator A can be considered a perturbation of the free, i.e. diagonal problem. When we look how A_{12} and A_{21} enter into (19) we expect to recover Liouville's formula, at least in a purely formal way, as we let them tend to zero. It is therefore reasonable to hope that we can bound the second trace on the right hand side by $\operatorname{tr} A_{12}A_{21}$. Fortunately, this can indeed be done with the aid of two elementary estimates on traces, which we prove for the sake of completeness and present in the order of their further appearance.

Lemma 3 *Let $L, M : \mathcal{H}_1 \to \mathcal{H}_1$ be bounded linear operators and additionally L be positive. Then,*

$$|\operatorname{tr} LM| \leq \|M\| \operatorname{tr} L.$$

PROOF. We transform L into diagonal form $L = WDW^*$ with a unitary matrix W and a diagonal matrix $D = \operatorname{diag}\{\lambda_1, \ldots, \lambda_N\}$ containing the eigenvalues of L. Then,

$$\operatorname{tr} LM = \operatorname{tr} WDW^*M = \operatorname{tr} DW^*MW = \sum_{j=1}^{N} \lambda_j (W^*MW)_{jj}.$$

Estimating yields

$$|\operatorname{tr} LM| \leq \sum_{j=1}^{N} \lambda_j |(W^*MW)_{jj}| \leq \|W^*MW\| \sum_{j=1}^{N} \lambda_j \leq \|M\| \operatorname{tr} L$$

where the positivity of L ensured $|\lambda_j| = \lambda_j$. This shows the statement. $\square$

Lemma 4 *Let $L : \mathcal{H}_2 \to \mathcal{H}_1$ and $M : \mathcal{H}_2 \to \mathcal{H}_2$ be bounded operators. Then,*

$$|\operatorname{tr} LML^*| \leq \|M\| \operatorname{tr} LL^*.$$

PROOF. In order to estimate the trace it suffices to estimate scalar products

$$|(\varphi, LML^*\varphi)| = |(L^*\varphi, ML^*\varphi)| \leq \|M\| \|L^*\varphi\|^2 = \|M\|(\varphi, LL^*\varphi).$$

This implies the lemma. $\square$

One might think of deducing the second lemma from the first by using the cyclic commutativity of the trace, $\operatorname{tr} LML^* = \operatorname{tr} ML^*L$, because L^*L is obviously positive. However, since $\mathcal{H}_2$ is infinitely dimensional this would have involved some deeper properties of trace class operators, which we did not want to use in order to keep the proofs as elementary as possible. Now everything is at hand to show the main theorem.

Theorem 5 *Let $U(t)$ be a semi-group of bounded operators with generator A that is bounded and self-adjoint. Define*

$$\gamma := \inf \sigma(A), \;\; \Gamma := \sup \sigma(A), \;\; \textit{and } C(T) := \int_0^T te^{(\Gamma-\gamma)t}\, dt.$$

Then, the section determinants admit the representation

$$\det U_{11}(T) = e^{T \operatorname{tr} A_{11}} e^{F(T)} \tag{20}$$

with the function F obeying

$$|F(T)| \leq C(T) \operatorname{tr} A_{12}A_{21}. \tag{21}$$

PROOF. We infer from (19) that in (20)

$$F(T) = \int_0^T \int_0^t E(t,\tau)\, d\tau\, dt$$

with

$$E(t,\tau) = \operatorname{tr} A_{12} e^{(t-\tau)A_{22}} A_{21} U_{11}(\tau) U_{11}(t)^{-1}.$$

Note that $A_{12}e^{(t-\tau)A_{22}}A_{21}$ is positive whence Lemma 3 yields

$$|E(t,\tau)| \le \|U_{11}(\tau)U_{11}(t)^{-1}\| \operatorname{tr} A_{12}e^{(t-\tau)A_{22}}A_{21}.$$

Since $A_{21} = A_{12}^*$ we may use Lemma 4

$$|E(t,\tau)| \le \|U_{11}(\tau)U_{11}(t)^{-1}\| \|e^{(t-\tau)A_{22}}\| \operatorname{tr} A_{12}A_{21}.$$

With the aid of Lemmas 1 and 2 we see

$$\begin{aligned} \|U_{11}(\tau)U_{11}(t)^{-1}\|\|e^{(t-\tau)A_{22}}\| \quad & \le \|U_{11}(\tau)\|\|U_{11}(t)^{-1}\|\|e^{(t-\tau)A_{22}}\| \\ & \le e^{\Gamma\tau}e^{-\gamma t}e^{\Gamma(t-\tau)} \\ & = e^{(\Gamma-\gamma)t}. \end{aligned}$$

Eventually, we obtain the bound

$$|F(T)| \le \int_0^T \int_0^t |E(t,\tau)|\, d\tau\, dt = \int_0^T te^{(\Gamma-\gamma)t}\, dt \operatorname{tr} A_{12}A_{21}.$$

This is the desired estimate. □

We briefly mark out where we can benefit from Theorem 5. Let us start with what we learn about how $\det U_{11}(T)$ depends on T. We easily infer from the definition that $\det U_{11}(T)$ can grow at most exponentially in T. Hence the function F in (20) ought to have a bound linearly in T whereas (21) yields an exponential bound which clearly does not properly reflect the long-time asymptotics at all. Unlike this, for $T \to 0$ (20) and (21) do give the accurate asymptotics but will not provide new insight. The actual value of Theorem 5 is the appearance of the term $\operatorname{tr} A_{12}A_{21}$ which allows us to study how the dimension N effects $\det U_{11}(T)$. In order to grasp what (21) actually means let us expand $\ln \det U_{11}(T)$ formally into a power series with respect to T around 0

$$\ln \det U_{11}(T) = T \operatorname{tr} A_{11} + \frac{T^2}{2} \operatorname{tr} A_{12}A_{21} + \frac{T^3}{6} \operatorname{tr}(A_{12}A_{22}A_{21} - A_{11}A_{12}A_{21}) + \cdots \tag{22}$$

We see that $\ln \det U_{11}(T) - T \operatorname{tr} A_{11}$ was estimated by the next term in the power series. This would be really difficult if we used the power series itself. Fortunately, the second term in (22) also appears in (19) though somehow interwoven with other quantities. We can see this as having summed up the power series to a form amenable to being estimated. It is tempting to go one step further namely to try to estimate

$$\ln\det U_{11}(T) - T\operatorname{tr} A_{11} - \frac{T^2}{2}\operatorname{tr} A_{12}A_{21}.$$

One could presume that this can be bounded by the third term in (22). However, the estimates used in Theorem 5 rely heavily on the positivity of $A_{12}A_{21}$ which cannot be assumed for $A_{12}A_{22}A_{21} - A_{11}A_{12}A_{21}$. Thus, our method breaks down at this point. A complete answer to this question would touch the celebrated problem of the third term.

Employing completely different techniques, in particular the Berezin-Lieb inequality, Laptev and Safarov [4] derived a variant of Theorem 5. Let A be written as $A = \psi(V)$ with a rather arbitrary function ψ then they could show

$$|\operatorname{tr}(A_{11} - \psi(V_{11})| \leq \frac{1}{2}\|\psi''\|_{L^\infty(K)} \operatorname{tr} V_{12}V_{21} \tag{23}$$

with some compact set K that is defined through the spectral data of V. For the special case $\psi(x) = \ln(x)$ this becomes

$$|F(1)| \leq \frac{1}{2}\|\psi''\|_{L^\infty(K)} \operatorname{tr} U_{12}U_{21} \tag{24}$$

Note the appearance of U_{12} and U_{21} in (24) instead of A_{12} and A_{21}. By dint of (17) we can transform this into (21). However, the constant that will appear in the inequality differs very much from ours because $\psi''(x) = 1/x^2$ whereas the dangerous part in $C(T)$ stems from $U_{11}(t)^{-1}$. If we want to generalize the results given herein to unbounded operators the respective constants both become infinity with $N \to \infty$ but $C(T)$ should not grow as fast as in the Laptev-Safarov estimate.

3 Szegő theorems

We are going to study the influence of the dimension N on the section determinants $\det U_{11}(T)$. In order to limit the number of indices we shall use orthogonal projections instead of operator block matrices. Let $P_N : \mathcal{H} \to \mathcal{H}$ be a sequence of orthogonal projections with finite dimensional range. It is not necessary to have the projections converge to the identity operator in the strong topology. The Hilbert space $\mathcal{H}$ is decomposed according to

$$\mathcal{H} = \operatorname{ran} P_N \oplus (\operatorname{ran} P_N)^\perp. \tag{25}$$

Thus, $\operatorname{ran} P_N$ plays the role of $\mathcal{H}_1$ of the foregoing section and $(\operatorname{ran} P_N)^\perp$ that of $\mathcal{H}_2$. The abstract Szegő theorem referred to in the title can now be stated.

Theorem 6 (Abstract Szegő Theorem) *Let $U : \mathcal{H} \to \mathcal{H}$ be a bounded, self-adjoint, strictly positive operator and put $A := \ln U$. Let $P_N : \mathcal{H} \to \mathcal{H}$ be a sequence of orthogonal projections with* $\dim \operatorname{ran} P_N = N \in \mathbb{N}$ *such that*

$$\operatorname{tr} P_N A(\mathbb{1} - P_N)AP_N = o(N) \text{ for } N \to \infty. \tag{26}$$

Then,

$$\lim_{N\to\infty} \left(e^{-\operatorname{tr} P_N A P_N} \det P_N U P_N \right)^{1/N} = 1. \tag{27}$$

Here the determinant $\det P_N U P_N$ *is to be understood as the determinant of the finite dimensional operator* $P_N U P_N : \operatorname{ran} P_N \to \operatorname{ran} P_N$.

PROOF. Since U is bounded and strictly positive we can define the logarithm $A := \ln U$ of U without any ado via the usual power-series and conclude $U = e^A$. The semi-group $U(t)$ needed to apply the estimate from Section 2 is simply defined by

$$U(t) := e^{tA},$$

which surely satisfies all the properties required for being a semi-group of bounded operators. If we identify U_{11} with $P_N U P_N$ etc. in an obvious way and note $U(1) = U$ then Theorem 5 with $T = 1$ yields

$$\det P_N U P_N = e^{\operatorname{tr} P_N A P_N} e^{F(1)}$$

where

$$|F(1)| \le C(1) \operatorname{tr} P_N A(\mathbb{1} - P_N)AP_N.$$

Having rewritten this as

$$\left(e^{-\operatorname{tr} P_N A P_N} \det P_N U P_N \right)^{1/N} = e^{\frac{1}{N} F(1)}$$

and letting N tend to infinity we deduce the assertion from the assumption on $\operatorname{tr} P_N A(\mathbb{1} - P_N)AP_N$. $\square$

We stress that Formula (27) cannot be rewritten as

$$\lim_{N\to\infty} (\det P_N U P_N)^{1/N} = \lim_{N\to\infty} \left(e^{\frac{1}{N} \operatorname{tr} P_N A P_N} \right).$$

For the limit on the right-hand side in general will not exist. To understand why let us represent the generator as infinite matrix $A = (a_{jk})_{j,k\in\mathbb{N}}$ whereby the exponent in question becomes

$$\frac{1}{N} \operatorname{tr} P_N A P_N = \frac{1}{N} \sum_{k=1}^{N} a_{kk} =: s_N.$$

Unfortunately, even the a_{kk} being bounded does not imply necessarily that the s_N converge. For instance, when we take

$$a_{kk} := k \sin \ln(k+1) - (k-1) \sin \ln k, \;\; k \in \mathbb{N},$$

then

$$s_N = \sin \ln(N+1).$$

Some reflections show the s_N do not converge though the a_{kk} are bounded. Nonetheless, in special cases of course it is possible to have convergence (see e.g. Theorem 10).

To derive Szegő theorems in concrete situations we have to check for which operators condition (26) is satisfied. This question is intimately related to the notion of Følner sequences (see [1]). Recall, that a sequence of finite dimensional projections P_N is called a Følner sequence for some operator A if $P_N \to \mathbb{1}$ strongly and

$$\lim_{N\to\infty} \frac{\operatorname{tr} P_N A(\mathbb{1} - P_N) A P_N}{\operatorname{tr} P_N} = 0. \tag{28}$$

The strong convergence has been included, contrary to our own assumptions, because such sequences play an important role in numerical analysis where approximation properties are of interest. The general problem in those applications is to find a Følner sequence for a given operator. In our context the projections are fixed and we ask for the class of those operators such that the projections are a Følner sequence for all these operators. Here we shall prove that the compact operators and the Toeplitz operators belong to this class whereas there are bounded operators that do not. As was mentioned in the introduction we shall not concentrate on further special cases to be treated but leave the more interesting problem of determining the general class of those operators for future investigations.

Let us start with the counterexample. As above we write $A = (a_{jk})_{j,k\in\mathbb{N}}$ whence

$$\operatorname{tr} P_N A(\mathbb{1} - P_N) A P_N = \sum_{j=1}^{N} \sum_{k=N+1}^{\infty} |a_{jk}|^2.$$

We now define for $j \leq k$

$$a_{jk} := \begin{cases} 1 & k = j^2, \\ 0 & otherwise, \end{cases}$$

and put $a_{kj} := a_{jk}$ which obviously induces a bounded self-adjoint operator A. Then

$$\begin{aligned}\operatorname{tr} P_N A(\mathbb{1} - P_N) A P_N &= \#\{j = 1, \ldots, N | j^2 \geq N + 1\} \\ &= N - \#\{j = 1, \ldots, N | j^2 \leq N\} \\ &= N - O(\sqrt{N})\end{aligned}$$

which implies

$$\frac{1}{N} \operatorname{tr} P_N A(\mathbb{1} - P_N) A P_N \to 1,\ N \to \infty.$$

We are thus faced with an bounded operator that does not satisfy (26). We turn to the case of compact operators.

Lemma 7 *Let $A : \mathcal{H} \to \mathcal{H}$ be a compact operator and $P_N : \mathcal{H} \to \mathcal{H}$ a sequence of finite dimensional orthogonal projections that converge strongly to the identity operator. Then,*

$$\lim_{N \to \infty} \frac{1}{N} \operatorname{tr} P_N A(\mathbb{1} - P_N) A P_N = 0.$$

PROOF. We recall from functional analysis that a compact operator turns strong convergence into operator convergence (see [3]). Hence, the strong convergence $\mathbb{1} - P_N \to 0$ implies the operator convergence $A(\mathbb{1} - P_N)A \to 0$. Since $P_N \geq 0$ we can apply Lemma 3 to obtain

$$\begin{aligned}|\operatorname{tr} P_N A(\mathbb{1} - P_N) A P_N| &= |\operatorname{tr} P_N A(\mathbb{1} - P_N) A| \\ &\leq \|A(\mathbb{1} - P_N) A\| \operatorname{tr} P_N \\ &= \|A(\mathbb{1} - P_N) A\| N.\end{aligned}$$

This proves the lemma. $\square$

The lemma shows that for each compact operator every sequence of orthogonal projections converging strongly to the identity is a Følner sequence for this operator. It can be generalized in a simple way by noting that $P_N A(\mathbb{1} - P_N) A P_N$ does not contain the diagonal elements of A and it is thus enough to require that A is the sum of a compact and a diagonal operator. For the sake of brevity, however, we shall not take this into account in the following weak Szegő theorem for compact operators, which we derive as a simple consequence of Lemma 7.

Theorem 8 *Let $U : \mathcal{H} \to \mathcal{H}$ be a bounded, self-adjoint, and strictly positive operator such that $A := \ln U$ is compact. Then,*

$$\lim_{N \to \infty} (e^{-\operatorname{tr} P_N A P_N} \det P_N U P_N)^{1/N} = 1.$$

PROOF. Combine Theorem 6 and Lemma 7. $\square$

Now we turn to Toeplitz operators. The underlying Hilbert space is chosen to be $\mathcal{H} = L^2[0, 2\pi]$. In $L^2[0, 2\pi]$ the exponentials $e_j(x) := \frac{1}{\sqrt{2\pi}} e^{ijx}$ form a complete orthonormal system. Let $u \in L^2[0, 2\pi]$ be a bounded real valued function with $\inf_{x \in [0,2\pi]} u(x) > 0$. We consider the multiplication operator U induced by u:

$$(U\varphi)(x) := u(x)\varphi(x). \tag{29}$$

U is a bounded, self-adjoint, and strictly positive operator. The latter is due to $u(x)$ not vanishing. We can therefore apply our general theory to this case. The logarithm $A := \ln U$ is well-defined and acts simply like

$$(A\varphi)(x) = a(x)\varphi(x) \tag{30}$$

with $a := \ln u$. The function a is real-valued and in $L^2[0, 2\pi]$ because u has no zeroes. We compute the matrix representation of U and A with respect to the basis formed by e_j. To this end denote by u_j and a_j the Fourier coefficients of u and a, respectively:

$$u_j := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-ijx} u(x)\, dx, \ \ j \in \mathbb{Z}. \tag{31}$$

We then have

$$(e_j, Ue_k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ijx} u(x) e^{ikx}\, dx = \frac{1}{2\pi} \int_0^{2\pi} e^{-i(j-k)x} u(x)\, dx = \frac{1}{\sqrt{2\pi}} u_{j-k}.$$

The matrix $U = (u_{j-k})_{j,k\in\mathbb{Z}}$ thus is of Toeplitz structure. The same holds for $A = (a_{j-k})_{j,k\in\mathbb{Z}}$. Note, that these are doubly sided infinite Toeplitz matrices. The generating sequences (u_j) and (a_j) are square-summable. In particular,

$$\sum_{j\in\mathbb{Z}} |a_j|^2 < \infty \tag{32}$$

because $a \in L^2[0, 2\pi]$. We take P_N to be the projection onto the subspace

$$\mathcal{H}_N := \operatorname{span}\{e_j | j = 1, \ldots, N\}. \tag{33}$$

Note, that the projections P_N do not converge strongly to $\mathbb{1}$ because we fixed the left 'corner' of $\mathcal{H}_N$. We did this because we like to have $N \times N$-Toeplitz matrices in order to be close to the classical theorem. We compute $\operatorname{tr} P_N A(\mathbb{1} - P_N)AP_N$. Recall that $a_{-k} = \bar{a}_k$ because a is real-valued.

$$\operatorname{tr} P_N A(\mathbb{1} - P_N)AP_N = \sum_{j=1}^{N} \left(\sum_{k=N+1}^{\infty} a_{j-k}a_{k-j} + \sum_{k=0}^{-\infty} a_{j-k}a_{k-j} \right) = 2\sum_{j=1}^{N}\sum_{k=j}^{\infty} |a_k|^2.$$

We split up this sum

$$\begin{aligned}\frac{1}{2}\operatorname{tr} P_N A(\mathbb{1}-P_N)AP_N &= \sum_{j=1}^{N}\sum_{k=j}^{N}|a_k|^2 + \sum_{j=1}^{N}\sum_{k=N+1}^{\infty}|a_k|^2 \\ &= \sum_{j=1}^{N} j|a_j|^2 + N\sum_{k=N+1}^{\infty}|a_k|^2.\end{aligned}$$

The second term is easily seen to be of order $o(N)$:

$$\frac{1}{2}\operatorname{tr} P_N A(\mathbb{1}-P_N)AP_N = \sum_{j=1}^{N} j|a_j|^2 + o(N). \tag{34}$$

The remaining term, which is a bit more dangerous, will be treated in the next lemma.

Lemma 9 *Let $c_n \geq 0$ a sequence of numbers such that the series $\sum_{j=1}^{\infty} c_j$ converges. Then,*

$$\sum_{j=1}^{N} jc_j = o(N),\ \ N\to\infty.$$

PROOF. We show that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N} jc_j = 0.$$

Let $\varepsilon > 0$ be given. We first pick $N_0 \in \mathbb{N}$ such that

$$\sum_{k=N_0}^{\infty} c_k \leq \frac{\varepsilon}{2}$$

which is possible because the series converges. For $N \geq N_0$ then,

$$\frac{1}{N}\sum_{k=1}^{N} kc_k = \frac{1}{N}\sum_{k=1}^{N_0} kc_k + \frac{1}{N}\sum_{k=N_0+1}^{N} kc_k \leq \frac{1}{N}\sum_{k=1}^{N_0} kc_k + \sum_{k=N_0+1}^{N} c_k \leq \frac{1}{N}\sum_{k=1}^{N_0} kc_k + \frac{\varepsilon}{2}.$$

Now N_0 is fixed whence we can make N such large that the remaining sum becomes smaller than $\varepsilon/2$. This shows the claimed convergence. □

We now recover Szegő's first theorem for Toeplitz matrices.

Theorem 10 (Szegő) *Let $u \in L^2[0,2\pi]$ with $\inf_{x\in[0,2\pi]} u(x) > 0$ and u_j be the Fourier coefficients of u according to* (31)*. Define furthermore $U_N =$*

$(u_{j-k})_{j,k=1,\ldots,N}$. *Then,*

$$\lim_{N\to\infty} (\det U_N)^{1/N} = \exp\left[\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \ln u(x)\, dx\right].$$

PROOF. First of all we see immediately that U_N is the matrix of the operator $P_N U P_N$: $\operatorname{ran} P_N \to \operatorname{ran} P_N$ with respect to the basis formed by the exponentials e_j. The trace $\operatorname{tr} P_N A P_N$ is computed explicitely

$$\operatorname{tr} P_N A P_N = \sum_{j=1}^{N} a_{j-j} = N a_0$$

and finally

$$a_0 = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} a(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \ln u(x)\, dx.$$

Along with (34), Lemma 9, and Theorem 6 this proves the lemma. □

Two problems arise cogently. Firstly, we want to push forward the asymptotic expansion and, hence, derive a second or strong Szegő theorem. Secondly, we liked to get rid of the strict positivity of the symbol u allowing u also to vanish somewhere which would make the generator A of the semi-group an unbounded yet semi-bounded operator. Surprisingly, (26) is still valid even then whereas the proof of Theorem 6 itself will no longer work.

We do not value the above result in the first place for having regained Szegő's classical theorem because what is already known to the literature exceeds Theorem 10 by far. The purpose of the abstract approach presented herein rather is on the one hand to see a special result from a general point of view and on the other hand to dispose of a method that can be flexibly applied to various situations that do not have as special features as Toeplitz matrices.

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