Integral Formulas for Section Determinants of Bounded Operators

PhD Thesis
by
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Contents

1 Introduction

The classical adiabatic theorems relate the spectral projections of self-adjoint operators to the solutions to time-dependent Schrödinger equations. In this thesis we present a new adiabatic theorem that expresses the section determinants of the spectral projections by means of the solution to a Wiener-Hopf equation.

Let A_0 be a self-adjoint and B be a bounded symmetric operator acting in a Hilbert space $\mathcal{H}$. We seek a connection between the spectral projections of A_0 and those of $A_0 + B$. To this end we introduce a parameter α that enables us to transform the two operators into each other:

$$A(\alpha) := A_0 + \alpha B, \ \ \alpha \in [0, 1]. \tag{1.1}$$

Let $P(\alpha)$ be the Riesz projection to a certain part of the spectrum of $A(\alpha)$. We shall prove in Theorem 4.1 the following integral formula that may be considered a variant of the adiabatic theorem:

$$\det P(0)P(1)P(0) = \lim_{\eta\to+0} \lim_{\varepsilon\to+0} \exp\left[-2\,\mathrm{Re}\int_0^1 \int_{-\infty}^0 e^{\eta t}\,\mathrm{tr}\, BG_\varepsilon(t, t+0; \alpha)\, dt\, d\alpha\right]. \tag{1.2}$$

The meaning of the determinant will be explained in Section 2.2. The operator $G_\varepsilon(t, t'; \alpha)$ is called the time-ordered Green operator. It satisfies the Wiener-Hopf equation ($\varphi \in \mathcal{H}$):

$$G_\varepsilon(t, t'; \alpha)\varphi = e^{-\varepsilon|t-t'|}G_0(t - t')\varphi + \alpha \int_{-\infty}^0 e^{-\varepsilon|t-\tau|}G_0(t - \tau)BG_\varepsilon(\tau, t'; \alpha)\varphi\, d\tau. \tag{1.3}$$

The operator G_0 is solely determined by A_0. Equations of this type were studied by Bart, Gohberg, and Kaashoek [4]. In Section 4.3 we shall independently develop a solution theory that focuses on the special right hand side $e^{-\varepsilon|t-t'|}G_0(t - t')$, thus making the calculations easier.

In order to better grasp the meaning of formula (1.2) and to justify the name 'adiabatic theorem' we quote the classical adiabatic theorem (see [2]). This is concerned with families of self-adjoint operators $H(s)$. We do not use the notation $A(\alpha)$ since the parameter s has a different meaning than α in (1.1). Moreover we stay in concordance with the physics literature. We call $H(s)$ Hamilton operator (see the physical motivation below). Let $P(s)$ be the spectral projection to a certain part of the spectrum of $H(s)$. Then $P(s)$ can be transformed asymptotically from $P(0)$, i.e., we have:

$$P(s) = U_\eta(s)P(0)U_\eta^*(s) + O(\eta), \ \ \eta \to 0, \tag{1.4}$$

where the transformation operators U_η satisfy a Schrödinger equation:

$$i\eta\frac{d}{ds}U_\eta(s) = H(s)U_\eta(s), \ \ U_\eta(0) = \mathbb{1}. \tag{1.5}$$

The variable s has the meaning of a scaled time which is related to the physical time t by $s = \eta t$. The Schrödinger equation can be transformed to physical time:

$$i\frac{d}{dt}\tilde{U}_\eta(t) = H(\eta t)\tilde{U}_\eta(t), \ \ \tilde{U}_\eta(0) = \mathbb{1}. \tag{1.6}$$

Physically the limit $\eta \to 0$ means that the Hamilton operator varies slowly in time. The intuitive idea behind the adiabatic theorem may be expressed as follows: The perturbation is switched on so carefully that at the end you think it has been switched on since ever.

Both theorems carry some similarities. The spectral projections cannot be computed directly but depend on the solutions to certain equations. The formulas are valid only asymptotically, i.e. to actually compute the spectral projections exactly some limits are to be performed. These limits are singular because neither the Wiener-Hopf equation nor the Schrödinger equation possess a well defined limit equation.

Beside these obvious similarities there is a deeper relationship between the two theorems because formula (1.2) can be derived at least formally from (1.4). In order to do this we need a formula for the section determinants of the solutions to Schrödinger equations. This is the second main result of this thesis. We shall prove it in Section 4. Let $U(t,\alpha)$ be the solution to

$$i\frac{d}{dt}U(t,\alpha) = A(\alpha)U(t,\alpha),\ U(0,\alpha) = \mathbb{1}. \tag{1.7}$$

Then for the section determinants the following formula holds:

$$\det PU(T,1)P = \det PU(T,0)P \exp\left[-\int_0^1\int_0^T \operatorname{tr} BG(t,t+0;\alpha)\,dt\,d\alpha\right] \tag{1.8}$$

where G satisfies the integral equation ($\varphi \in \mathcal{H}$):

$$G(t,t';\alpha)\varphi = G_0(t-t')\varphi + \alpha\int_0^T G_0(t-\tau)BG(\tau,t';\alpha)\varphi\,d\tau. \tag{1.9}$$

Such convolution equations on finite intervalls were studied by Bart, Gohberg, and Kaashoek [3]. We shall not use their results but present a solution theory in Section 4.3 that concentrates completely on the special right hand side $G_0(t-t')$, thus avoiding superfluous calculations.

Section determinants as in (1.8) have been the subject to various investigations. For the first time Szegö [16] studied the section determinants of Toeplitz matrices. Therefore theorems concerning those determinants are usually called Szegö theorems. The first abstract result goes back to Widom [17] who considered operators of the form Pe^CP where C is a trace class operator. He applied his result to integral operators where P projects onto a domain in $\mathbb{R}^d$. By making this domain larger and larger he could study the differential geometric properties of the domain of integration. This gives a connection to spectral geometry. Dym and Ta'assan [7] could allow bounded operators C. The main tools in both papers are factorization theorems based upon a general addition theorem for operator valued exponential functions known as Baker-Campbell-Hausdorff formula. Widom's question whether the determinant can be studied without using this formula may be answered with the aid of (1.8) by carefully analyzing the integral equation (1.9) (see Section 4.3). Contrary to these papers, our approach allows unbounded operators what is important for applications in quantum mechanics.

In order to apply the integral formula to the adiabatic theorem we need a generalization of (1.8) (see Section 4.4) which we present without giving a proof but which can be treated by our methods. Let $U_\eta(t,\alpha)$ be the solution to:

$$i\frac{d}{dt}U_\eta(t,\alpha) = (A_0 + \alpha e^{\eta t}B)U_\eta(t,\alpha),\ \lim_{t\to-\infty} U(-t,0)U_\eta(t,\alpha) = \mathbb{1},\ U(0,0) = \mathbb{1}. \tag{1.10}$$

Physically spoken, this means that the perturbation has been switched on an infinitly long time ago. Therefore we had to impose the initial condition at $t = -\infty$. We obtain:

$$\det PU_\eta(0,1)P = \det PU(0,0)P \exp\left[-\int_0^1 \int_{-\infty}^0 e^{\eta t} \operatorname{tr} BG^\eta(t,t+0;\alpha)\,dt\,d\alpha\right] \qquad (1.11)$$

where G^η satisfies the integral equation ($\varphi \in \mathcal{H}$):

$$G^\eta(t,t';\alpha)\varphi = G_0(t-t')\varphi + \alpha \int_{-\infty}^0 e^{\eta\tau} G_0(t-\tau) BG^\eta(\tau,t';\alpha)\varphi\,d\tau. \qquad (1.12)$$

Formula (1.11) appears for the first time in Rivier und Simanek [15] who indicate only a formal derivation using infinite series. Because of the factor $e^{\eta\tau}$ (1.12) is not a convolution equation. It is tempting to perform the limit $\eta \to 0$ in (1.12). But the limit equation is not well defined. Therefore we introduce a second parameter $\varepsilon > 0$:

$$G^\eta_\varepsilon(t,t';\alpha)\varphi = e^{-\varepsilon|t-t'|} G_0(t-t')\varphi + \alpha \int_{-\infty}^0 e^{\eta\tau} e^{-\varepsilon|t-\tau|} G_0(t-\tau) BG^\eta_\varepsilon(\tau,t';\alpha)\varphi\,d\tau. \qquad (1.13)$$

Now we can perform the limit $\eta \to 0$ in (1.13). The question is do the solutions converge for $\eta \to 0$ and if they do so, do they converge to the solution to the limit equation? In this way we have recovered our adiabatic theorem. The parameter η in (1.2) may be interpreted as a kind of switching parameter. In the physics literature the parameter ε only enters implicitly via the Fourier transform. We do not wish to make this derivation rigorous because the proof in Section 5.2 does not need the classical adiabatic theorem.

We return to our adiabatic theorem. Having considered so far the similarities we now look at the differences. First of all we have different types of equations. On the one hand there is a Wiener-Hopf equation, on the other hand there is a Schrödinger equation although G_ε at least formally satisfies a differential equation, namely:

$$(i\frac{\partial}{\partial t} - A(\alpha))G_\varepsilon(t,t';\alpha) = \delta(t-t')\mathbb{1}.$$

The more essential difference is provided by the two parameters in (1.2) contrary to the one parameter in (1.4). As we shall see in Section 4, these parameters are connected with different assumptions. The parameter η in both theorems requires a gap in the spectrum. This once again justifies the corresponding names. But the parameter ε requires a gap in the numerical range of $A(\alpha)$. This is needed in order to assure that a certain operator has no real spectral values. Such operators are called dichotomous. Thus we have arrived at the third main result of this thesis. Because the known dichotomy criterion of Baskakov does not suffice for our purposes we prove a new dichotomy criterion in Section 3.2.

As was suggested several times the problems under consideration may be motivated physically, as will be made more precise now. In quantum mechanics a physical system is described by a Hilbert space $\mathcal{H}$, the normed vectors of which represent the states of the system. The probability that the system goes from a state φ into a state ψ is given by $|(\varphi,\psi)|^2$. An observable is represented by a self-adjoint operator A on $\mathcal{H}$. The possible measure values are contained in the spectrum of A. An orthogonal projection onto a vector φ can be interpreted as measurement whether the system is in the state φ or not. A particular important observable is the total energy. The corresponding operator is called Hamilton operator. It characterizes the physical system and governs the time evolution of the system via the Schrödinger equation:

$$i\frac{d}{dt}U(t)\varphi = HU(t)\varphi, U(0) = \mathbb{1},\ \varphi \in \mathcal{H}.$$

The unitary operator $U(t)$ is called time evolution operator. It describes how the states change in time.

We pose the question with what probability the system returns into its original state. To be more precise: Let the system start in the state φ and evolve according to the Schrödinger equation up to time t. Then it is in the state $U(t)\varphi$. What is the probability to measure the state φ? By the above considerations we have to study the scalar product $(\varphi, U(t)\varphi)$.

We consider two Hamilton operators namely the unperturbed operator H_0 and the perturbed operator $H := H_0 + V$. We interpret the transition from H_0 to H as suddenly switching on a perturbing potential V.[1] One may think for example of electrons and a magnetic field which is spontaneously switched on. Let the operators H_0 and H have a purely discrete spectrum with corresponding eigenvectors φ_j and ψ_j, $j \in \mathbb{N}$. We ask how likely it is that the system goes from the state φ_j into the state ψ_k when the perturbation is switched on, i.e., what is the value of $|(\varphi_j, \psi_k)|^2$? Since the Hamilton operator represents the total energy in particular the question arises whether the system goes from a state of lowest energy into such a state.

So far we have talked about quantum mechanical systems in general. We now turn to many-particle physics, especially to systems consisting of N identical fermions.[2] Let a single fermion be described by a Hilbert space $\mathcal{H}_1$, the one particle space. According to general principles the Hilbert space $\mathcal{H}_N$ called the N-particle space of a system consisting of N fermions is given by the antisymmetric tensorproduct:

$$\mathcal{H}_N := \underbrace{\mathcal{H}_1 \wedge \ldots \wedge \mathcal{H}_1}_{N\text{-times}}.$$

Scalarproducts in N-particle space can be expressed by scalarproducts in 1-particle space:

$$(\varphi_1 \wedge \ldots \wedge \varphi_N, \psi_1 \wedge \ldots \psi_N) = \det((\varphi_j, \psi_k))_{j,k=1,\ldots,N}.$$

We suppose that the particles do not interact.[3] Then we can express the N-particle Hamilton operator H_N and the time evolution operator $U_N(t)$ by the corresponding 1-particle operators H_1 and $U_1(t)$:

$$H_N = H_1 \wedge \mathbb{1} \wedge \ldots \wedge \mathbb{1} + \cdots + \mathbb{1} \wedge \ldots \wedge \mathbb{1} \wedge H_1, \; U_N(t) = U_1(t) \wedge \ldots \wedge U_1(t).$$

The formula for the Hamilton operator reflects the fact that without interaction the energies of the single particles are simply summed up to the total energy.

The above questions about certain probabilities had led us to different scalarproducts. In the case of an N-particle system these can be reduced to quantities in the 1-particle space:

$$\begin{aligned}(\varphi^N, U_N(t)\varphi^N) = (\varphi_1 \wedge \ldots \wedge \varphi_N, U(t)\varphi_1 \wedge \ldots \wedge U(t)\varphi_N) &= \det((\varphi_j, U(t)\varphi_k)_{j,k=1,\ldots,N}) \\ &= \det P_N U(t) P_N.\end{aligned}$$

Here P_N is the orthogonal projection onto $\operatorname{span}\{\varphi_1, \ldots, \varphi_N\}$. The determinant is to be understood as determinant of the finite dimensional operator $P_N U(t) P_N : \operatorname{ran} P_N \to \operatorname{ran} P_N$.

[1] Of course this is an ideal picture because any real switching process needs some time.

[2] Fermions are particles with half-integer spin what is not important in what follows. The only thing that counts is the antisymmetric tensor product. Contrary to that particles with integer spin or bosons are related to symmetric tensor product.

[3] This is approximately realized for electrons in a crystal lattice.

The last equality easily follows from

$$P_N = \sum_{j=1}^{N} (\varphi_j, \cdot)\varphi_j.$$

Analogously one obtains the following formula (see Section 2.2):

$$|(\varphi^N, \psi^N)|^2 = |(\varphi_1 \wedge \ldots \wedge \varphi_N, \psi_1 \wedge \ldots \wedge \psi_N)|^2 = |\det((\varphi_j, \psi_k)_{j,k=1,\ldots,N})|^2 = \det P_N \tilde{P}_N P_N$$

$\tilde{P}_N$ being the orthogonal projection onto $\operatorname{span}\{\psi_1, \ldots, \psi_N\}$.

The determinant $\det P_N \tilde{P}_N P_N$ plays an important role in physics at the so called X-ray-edge effect and was studied for the first time by Anderson [1]. The problem is that physically one is interested in free particles moving in $\mathbb{R}^d$ which are exposed suddenly to a perturbation potential. In this case the spectrum of the Hamilton operator is continuous thus, making our adiabatic theorem, which requires some gap conditions, inapplicable. If the particles are only allowed to move in a finite domain with size L some properties of the determinant may derived. Anderson derived an inequality by using asymptotic formulas for large N. But he neglected the error terms depending on the size L according to $O(L)$ for large L. Since then his result is known in physics as Anderson's orthogonality theorem or orthogonality catastrophe. Rivier and Simanek [15] and Hamann [8] used the classical adiabatic theorem and formula (1.11) for a special V in order to obtain the exact asymptotics for large particle numbers N. These papers contain some mathematical errors. Thus the results are to be taken unproven.[4] Yamada and Yosida [18] tried to treat general potentials. They based their computations upon an incorrect method for solving the Wiener-Hopf equation. The modification given by them later (see [19]) cannot be understood mathematically. The exact analysis of the section determinants especially for large N therefore must be considered an open problem for which the methods developed in this thesis can lead to a rigorous treatment.

The thesis is organized as follows. In Chapter 2 we sum up the tools necessary for our considerations. Chapter 3 provides the dichotomy criterion that will be needed in proving the integral formula for Riesz projections. The latter is proved in Chapter 5. We have put the integral formula for section determinants of groups of bounded operators before that because the computational idea is transparent already here. In the last Chapter 6 we do somewhat quantitative considerations concerning unitary operators and spectral projections.[5]

At this point I wish to thank Professor Kress for supervising this thesis and Professor Schönhammer for pointing out this interesting questions to me.

[4] Nevertheless in the physics literature these results are widely accepted.

[5] This chapter must be considered out-dated because in the meantime the author has derived much more exact and general criteria.

2 Preliminaries

2.1 General remarks

Henceforth $\mathcal{H}$ always denotes a separable complex Hilbert space. If $\mathcal{H}$ is finite dimensional this will be mentioned explicitly. The scalar product in $\mathcal{H}$ is linear in the second argument. All operators that will appear are linear without pointing this out. $D(S) \subset \mathcal{H}$ denotes the domain of definition of the operator $S : D(S) \to \mathcal{H}$. We always assume $D(S)$ to be dense in $\mathcal{H}$. $\operatorname{ran} S$ is the range of S.

We recall some simple facts concerning the analysis in Hilbert spaces. For details we refer to [9]. Continuity of functions having values in a Hilbert space is defined in the usual way via the norm. For operator-valued functions there are different possibilities to introduce continuity. If the operators are bounded the definition can be based on the operator norm. We then call the function continuous in the operator norm. Another way is to consider for fixed $\varphi \in \mathcal{H}$ the function $\varphi(\cdot) : x \mapsto S(x)\varphi$ and define continuity via the Hilbert space norm. We then call $S(\cdot)$ strongly continuous.

The above remarks carry over to the differentiation of vector-valued or operator-valued functions. The usual rules are valid in the general case if we notice that the operators involved do not commute in general.

Integrals are always to be understood as Riemann integrals. We again distinguish between integrals in the operator norm and strong integrals. Complex integrals along a path $\Gamma \subset \mathbb{C}$ are defined as usual. A closed, sufficiently smooth, and positively oriented path in the complex plane is for short called integration path or just path. We shall often need the standard estimate for integrals:

$$\| \int_\Gamma F(x)\, dx \| \leq |\Gamma| \sup_{x \in \Gamma} \|F(x)\|.$$

Here $|\Gamma|$ is the length of the integration contour being an interval in $\mathbb{R}$ or an path in $\mathbb{C}$. F is a bounded, integrable, vector- or operator-valued function. Moreover for bounded operators $S : \mathcal{H} \to \mathcal{H}$ we have:

$$\int_\Gamma SF(x)\, dx = S \int_\Gamma F(x)\, dx.$$

This may be incorrect for general operators.

2.2 Some linear algebra

We recall some simple properties of trace class operators (see [12]). A bounded linear operator $S : \mathcal{H} \to \mathcal{H}$ is called trace class operator if there is a complete orthonormal system $\{\varphi_j\}$ such that the series

$$\operatorname{tr} S := \sum_{j=1}^{\infty} (\varphi_j, S\varphi_j)$$

converges absolutely. In this case the series converges absolutely for every orthonormal system and the limit does not depend on the choice of the orthonormal system. $\operatorname{tr} S$ is called the trace of S.

Theorem 2.1. *Let $S_1 : \mathcal{H} \to \mathcal{H}$ be a trace class operator and $S_2 : \mathcal{H} \to \mathcal{H}$ be bounded. Then $S_1 S_2$ and $S_2 S_1$ are also trace class and we have:*

$$\operatorname{tr} S_1 S_2 = \operatorname{tr} S_2 S_1.$$

Proof. See [12]. □

Theorem 2.2. *Let $S : \mathcal{H} \to \mathcal{H}$ be trace class. Then the adjoint operator $S^* : \mathcal{H} \to \mathcal{H}$ is trace class, too, and we have:*

$$\operatorname{tr} S^* = (\operatorname{tr} S)^*.$$

Here * on the right hand side means complex conjugation.

Proof. We have:

$$\operatorname{tr} S^* = \sum_{j=1}^{\infty} (\varphi_j, S^* \varphi_j) = \sum_{j=1}^{\infty} (S\varphi_j, \varphi_j) = \sum_{j=1}^{\infty} (\varphi_j, S\varphi_j)^* = (\operatorname{tr} S)^*.$$

This proves the theorem. □

Theorem 2.3. *Let $P : \mathcal{H} \to \mathcal{H}$ be an orthonormal projection having an N-dimensional range. Then:*

1. *P is trace class.*
2. *Let $S : \mathcal{H} \to \mathcal{H}$ be bounded. Then:*

$$|\operatorname{tr} PS| \leq N \|S\|.$$

3. *Let $S(\beta) : \mathcal{H} \to \mathcal{H}$ be a family of bounded operators depending strongly continuously on $\beta \in \mathbb{R}$.[1] Then the map $\beta \mapsto \operatorname{tr} PS(\beta)$ is continuous.*

Proof. There is an orthonormal system $\{\varphi_1, \dots, \varphi_N\} \subset \mathcal{H}$ with:

$$P = \sum_{j=1}^{N} (\varphi_j, \cdot) \varphi_j.$$

For computing the trace it suffices to use the orthonormal system $\varphi_1, \dots, \varphi_N$:

$$\operatorname{tr} P = \sum_{j=1}^{N} \|\varphi_j\|^2 = N < \infty.$$

[1] In Sections 2.2, 2.3, and 2.4 we use β for the parameter of operator families in order to avoid confusion with the several parameters in Section 5.

This shows 1. 2. follows analogously:

$$|\operatorname{tr} PS| = |\sum_{j=1}^{N}(\varphi_j, S\varphi_j)| \leq \sum_{j=1}^{N} \|S\|\|\varphi_j\|^2 = N\|S\|.$$

Finally we have:

$$\operatorname{tr} PS(\beta) = \sum_{j=1}^{N}(\varphi_j, S(\beta)\varphi_j).$$

The map $\beta \mapsto (\varphi_j, S(\beta)\varphi_j)$ is continuous. Thus $\operatorname{tr} PS(\beta)$ is continuous as a finite sum of continuous functions. This proves 3. □

We define the determinant of finite dimensional operators as usually as normed, alternating, multilinear form on the column vectors of the corresponding matrix. It is well known from linear algebra that this definition does not depend on the choice of the basis. There is a simple but important relation between determinant and trace, which will be the fundament of Sections 4 and 5.

Theorem 2.4. *Let $\mathcal{H}$ be a finite dimensional Hilbert space and $S(\beta) : \mathcal{H} \to \mathcal{H}$, $\beta_0 \leq \beta \leq \beta_1$, be a family of invertible operators depending continuously differentiable on β. Then the following formula holds:*

$$\frac{d}{d\beta} \ln \det S(\beta) = \operatorname{tr}(S^{-1}(\beta)S'(\beta)).$$

Here $'$ denotes the derivative with respect to β.

Proof. Let $N := \dim \mathcal{H}$. Let $s_1(\beta), \ldots, s_N(\beta)$ the column vectors of $S(\beta)$. Then:

$$\frac{d}{d\beta} \det S(\beta) = \sum_{k=1}^{N} \det(s_1(\beta), \ldots, s_{k-1}(\beta), s_k'(\beta), s_{k+1}(\beta), \ldots, s_N(\beta)).$$

Let s_{jk} be the entries of S. We expand the k-th summand by the k-th column:

$$\det(s_1, \ldots, s_{k-1}, s_k', s_{k+1}, \ldots, s_N) = \sum_{j=1}^{N}(-1)^{j+k} D_{jk} s_{jk}'.$$

Here D_{jk} is the jk-th minor of S. We obtain:

$$\frac{d}{d\beta} \det S(\beta) = \sum_{k=1}^{N}\sum_{j=1}^{N}(-1)^{j+k} D_{jk}(\beta) s_{jk}'(\beta) = \operatorname{tr} D(\beta)S'(\beta)$$

with $D(\beta) := ((-1)^{j+k} D_{kj}(\beta))_{j,k=1,\ldots,N}$. Cramer's rule shows $D(\beta) = \det(S(\beta)) \cdot S^{-1}(\beta)$. This proves the theorem. □

We wish to integrate the formula in Theorem 2.4. To this end we need the continuity of $S^{-1}(\beta)$.

Theorem 2.5. *Let $S(\beta) : \mathcal{H} \to \mathcal{H}$, $\beta_0 \leq \beta \leq \beta_1$, be a family of invertible bounded operators depending continuously in the operator norm on β. Then $S^{-1}(\beta)$ is also bounded and depends continuously in the operator norm on β.*

Proof. By the homeomorphy theorem $S^{-1}(\beta)$ is bounded. We show the continuous dependence. We have:

$$S(\beta+h) = S(\beta)(\mathbb{1} + S^{-1}(\beta)(S(\beta+h) - S(\beta))).$$

Pick $0 < C < 1$. $S(\cdot)$ is continuous. Thus there is a $h_0 > 0$ such that

$$\|S^{-1}(\beta)\| \|S(\beta+h) - S(\beta)\| \leq C < 1$$

for all h with $|h| \leq h_0$. The Neumann series gives the estimate:

$$\|S(\beta+h)^{-1}\| \leq \frac{\|S^{-1}(\beta)\|}{1-C}.$$

We have furthermore:

$$\begin{aligned}\|S^{-1}(\beta+h) - S^{-1}(\beta)\| &\leq \|S^{-1}(\beta+h)\| \|S(\beta) - S(\beta+h)\| \|S^{-1}(\beta)\| \\ &\leq \frac{\|S^{-1}(\beta)\|^2}{1-C} \|S(\beta+h) - S(\beta)\|.\end{aligned}$$

This shows the continuity. □

We can now rewrite Theorem 2.4 in a trivial way.

Corollary 2.6. *Under the assumptions of Theorem 2.4 we have:*

$$\det S(\beta_1) = \det S(\beta_0) \exp\left[\int_{\beta_0}^{\beta_1} \operatorname{tr} S^{-1}(\beta) S'(\beta)\, d\beta\right].$$

Proof. Integrating and exponentiating in Theorem 2.4. □

Like the trace the definition of the determinant can be extended to arbitrary separable Hilbert spaces. In what follows we only need a special case namely the determinant of operators having the form PSP where P is the orthogonal projection onto a finite dimensional subspace. We simply define $\det PSP$ to be:[2]

$$\det PSP := \det(PSP : \operatorname{ran} P \to \operatorname{ran} P).$$

In what follows all determinants of operators having the form PSP are always meant in this sense.

Let $\{\varphi_j\} \subset \mathcal{H}$ and $\{\psi_j\} \subset \mathcal{H}$ be orthonormal systems. We consider the determinant:

$$\Delta_N := \det \begin{pmatrix} (\varphi_1, \psi_1) & \cdots & (\varphi_1, \psi_N) \\ \vdots & & \vdots \\ (\varphi_N, \psi_1) & \cdots & (\varphi_N, \psi_N) \end{pmatrix}. \tag{2.1}$$

Δ_N can be brought into another form more suitable for analytical treatment.

[2] In $PSP : \operatorname{ran} P \to \operatorname{ran} P$ we could of course replace PSP by PS. Since we often consider operators both on $\operatorname{ran} P$ and on the whole of $\mathcal{H}$ we always use the notation PSP.

Lemma 2.7. *Let $\{\varphi_j\}_{j\in\mathbb{N}}$, $\{\psi_j\}_{j\in\mathbb{N}} \subset \mathcal{H}$ be orthonormal systems, P_N be the orthogonal projection onto* $\operatorname{span}\{\varphi_1, \dots, \varphi_N\}$*, and $\tilde{P}_N$ be the orthogonal projection onto* $\operatorname{span}\{\psi_1, \dots, \psi_N\}$*. Then Δ_N from (2.1) can be written as:*

$$|\Delta_N|^2 = \det P_N \tilde{P}_N P_N.$$

Proof. We put $Z := ((\varphi_j, \psi_k))_{j,k=1,\dots,N}$. By the multiplication theorem for determinants we have:

$$|\det Z|^2 = \det Z (\det Z)^* = \det Z \det Z^* = \det Z Z^*.$$

We compute ZZ^*:

$$(ZZ^*)_{j,k} = \sum_{l=1}^{N} (\varphi_j, \psi_l)(\psi_l, \varphi_k) = \left(\varphi_j, \sum_{l=1}^{N} (\psi_l, \varphi_k)\psi_l\right).$$

The uniquely determined orthogonal projection onto $\operatorname{span}\{\psi_1, \dots, \psi_N\}$ has the form:

$$\tilde{P}_N = \sum_{l=1}^{N} (\psi_l, \cdot)\psi_l.$$

Thus we obtain:

$$(ZZ^*)_{jk} = (\varphi_j, \tilde{P}_N \varphi_k).$$

These are the matrix elements of $\tilde{P}_N$ in the basis $\{\varphi_j\}$. This proves the statement. □

In the context of the integral formulas in Sections 4 and 5 we need the inverse of an operator with respect to a subspace.

Theorem 2.8. *Let $S : \mathcal{H} \to \mathcal{H}$ be a linear operator and $P : \mathcal{H} \to \mathcal{H}$ an orthogonal projection such that the operator $(PSP|_{\operatorname{ran} P})^{-1} : \operatorname{ran} P \to \operatorname{ran} P$ exists. We define the operator $S^+ : \mathcal{H} \to \mathcal{H}$ as follows. First we define*

$$S^+\varphi := \begin{cases} (PSP|_{\operatorname{ran} P})^{-1}\varphi & \textit{for } \varphi \in \operatorname{ran} P, \\ 0 & \textit{for } \varphi \in (\operatorname{ran} P)^\perp \end{cases}$$

and then extend S^+ by linearity to the whole of $\mathcal{H}$. Then we have $S^+ = S^+P = PS^+$ and $S^+SP = PSS^+ = P$. If S and P commute we have $SS^+ = S^+S = P$. We call S^+ pseudo inverse of S with respect to P.

Proof. We have:

$$S^+\varphi = S^+ \underbrace{P\varphi}_{\in \operatorname{ran} P} + S^+ \underbrace{(\mathbb{1} - P)\varphi}_{\in \operatorname{ran}(\mathbb{1}-P)} = S^+P\varphi + 0.$$

Thus $S^+ = S^+P$. Since $S^+ : \operatorname{ran} P \to \operatorname{ran} P$ it follows $S^+P\varphi \in \operatorname{ran} P$ and hence $S^+P\varphi = PS^+P\varphi$. This gives $S^+ = PS^+$.

Moreover we have:

$$S^+SP\varphi = S^+PSP\varphi + S^+(\mathbb{1} - P)SP\varphi = P\varphi + 0$$

and by the first part:

$$PSS^+\varphi = PSPS^+\varphi = P\varphi.$$

If S and P commute it follows from what has been proved so far:

$$SS^+ = SPS^+ = PSS^+ = P$$
$$S^+S = SPS = S^+SP = P.$$

This completes the proof. $\square$

We rewrite the determinant formula in Corollary 2.6 for orthogonal projections.

Lemma 2.9. *Let $P(\beta) : \mathcal{H} \to \mathcal{H}$, $0 \leq \beta \leq 1$, be a family of orthogonal projections depending continuously differentiable in the operator norm on β. Moreover let $P : \mathcal{H} \to \mathcal{H}$ be an orthogonal projection having finite dimensional range and the property* $\det PP(\beta)P \neq 0$ *for all $\beta \in [0,1]$. Then:*

$$\det PP(1)P = \det PP(0)P \exp\left[2\,\mathrm{Re}\int_0^1 \mathrm{tr}\, P^+(\beta)(\mathbb{1} - P(\beta))P'(\beta)P(\beta)\, d\beta\right]. \tag{2.2}$$

Here $P^+(\beta)$ is the pseudo inverse of $P(\beta)$ with respect to P according to Theorem 2.8.

Proof. Since $P(\beta)$ is continuously differentiable with respect to β and $\det PP(\beta)P \neq 0$ we can apply Corollary 2.6 to the finite dimensional operator $PP(\beta)P : \mathrm{ran}\, P \to \mathrm{ran}\, P$:

$$\det PP(1)P = \det PP(0)P \exp\left[\int_0^1 \mathrm{tr}\, P^+(\beta)P'(\beta)\, d\beta\right].$$

Here we have noticed $P^+(\beta) = PP(\beta)$ (see Theorem 2.8). Differentiating the identity $P(\beta) = P(\beta)^2$ yields:

$$P'(\beta) = P(\beta)P'(\beta) + P'(\beta)P(\beta). \tag{2.3}$$

$P^+(\beta)$ and $P'(\beta)$ are self-adjoint because $P(\beta)$ is. We notice $\mathrm{tr}\, S^* = (\mathrm{tr}\, S)^*$ (see 2.2) and calculate:

$$\begin{aligned}
\mathrm{tr}\, P^+(\beta)P'(\beta) &= \mathrm{tr}\, P^+(\beta)[P(\beta)P'(\beta) + P'(\beta)P(\beta)] \\
&= \mathrm{tr}\, P^+(\beta)P(\beta)P'(\beta) + \mathrm{tr}\, P^+(\beta)P'(\beta)P(\beta) \\
&= (\mathrm{tr}\, P'(\beta)P(\beta)P^+(\beta))^* + \mathrm{tr}\, P^+(\beta)P'(\beta)P(\beta) \\
&= (\mathrm{tr}\, P^+(\beta)P'(\beta)P(\beta))^* + \mathrm{tr}\, P^+(\beta)P'(\beta)P(\beta) \\
&= 2\,\mathrm{Re}\,\mathrm{tr}\, P^+(\beta)P'(\beta)P(\beta)
\end{aligned}$$

where we have used $P(\beta)^2 = P(\beta)$ several times. We multiply equation (2.3) by $P(\beta)$ and obtain:

$$P(\beta)P'(\beta)P(\beta) = 0.$$

This gives:

$$P'(\beta)P(\beta) = (\mathbb{1} - P(\beta))P'(\beta)P(\beta).$$

This shows the statement. $\square$

We need a statement whether projections behave continuously.

Lemma 2.10. *Let $P(\beta)$ be a family of projections depending contiuously on β. Then* $\dim \mathrm{ran}\, P(\beta) = $ *const.*

Proof. [10], Lemma I.4.10 and Problem III.3.31. $\square$

2.3 Riesz projections

The resolvent set $\rho(S)$ of a closed and densely defined operator $S : D(S) \to \mathcal{H}$ is defined to be the set of all $z \in \mathbb{C}$ for which the operator $(z\mathbb{1} - S)^{-1} : \mathcal{H} \to D(S)$ called the resolvent exists and is bounded.[3] The set $\sigma(S) := \mathbb{C} \setminus \rho(S)$ is called the spectrum of S. We examine how $\rho(S)$ depends on z and S.

Theorem 2.11. *Let $S_0 : D(S_0) \to \mathcal{H}$ be a closed operator and $z_0 \in \rho(S_0)$. Let $S_1 : \mathcal{H} \to \mathcal{H}$ be a bounded operator and $z \in \mathbb{C}$ having the property*

$$(|z - z_0| + \|S_1\|)\|(z_0\mathbb{1} - S_0)^{-1}\| < 1. \tag{2.4}$$

We put $S := S_0 + S_1$. Then $z \in \rho(S)$ and we have:

$$\|(z\mathbb{1} - S)^{-1} - (z_0\mathbb{1} - S_0)^{-1}\| \leq (|z - z_0| + \|S_1\|)\frac{\|(z_0\mathbb{1} - S_0)^{-1}\|^2}{1 - (|z - z_0| + \|S_1\|)\|(z_0\mathbb{1} - S_0)^{-1}\|}. \tag{2.5}$$

Proof. By assumption $(z_0\mathbb{1} - S_0)^{-1} : \mathcal{H} \to \mathcal{H}$ exists and is bounded. We compute:

$$z\mathbb{1} - S = z_0\mathbb{1} - S_0 + (z - z_0)\mathbb{1} - S_1 = (z_0\mathbb{1} - S_0)(\mathbb{1} + (z_0\mathbb{1} - S_0)^{-1}((z - z_0)\mathbb{1} - S_1)).$$

(2.4) implies:

$$\|(z_0\mathbb{1} - S_0)^{-1}((z - z_0)\mathbb{1} - S_1)\| \leq \|(z_0\mathbb{1} - S_0)^{-1}\|(|z - z_0| + \|S_1\|) < 1.$$

The Neumann series shows that $(z\mathbb{1} - S)^{-1} : \mathcal{H} \to \mathcal{H}$ exists and is bounded with bound:

$$\|(z\mathbb{1} - S)^{-1}\| \leq \frac{\|(z_0\mathbb{1} - S_0)^{-1}\|}{1 - \|(z_0\mathbb{1} - S_0)^{-1}\|(|z - z_0| + \|S_1\|)}. \tag{2.6}$$

Thus $z \in \rho(S)$. From (2.6) and the formula

$$(z\mathbb{1} - S)^{-1} - (z_0\mathbb{1} - S_0)^{-1} = (z\mathbb{1} - S)^{-1}(S_1 + (z_0 - z)\mathbb{1})(z_0\mathbb{1} - S_0)^{-1}$$

we obtain the estimate (2.5):

$$\begin{aligned}\|(z\mathbb{1} - S)^{-1} - (z_0\mathbb{1} - S_0)^{-1}\| &\leq (|z - z_0| + \|S_1\|)\|(z_0\mathbb{1} - S_0)^{-1}\|\|(z\mathbb{1} - S)^{-1}\| \\ &\leq (|z - z_0| + \|S_1\|)\frac{\|(z_0\mathbb{1} - S_0)^{-1}\|^2}{1 - \|(z_0\mathbb{1} - S_0)^{-1}\|(|z - z_0| + \|S_1\|)}.\end{aligned}$$

The proof is finished. $\square$

This theorem implies that the resolvent set is open and therefore the spectrum is closed. The map $z \mapsto (z\mathbb{1} - S)^{-1}$ is continuous on $\rho(S)$ in the operator norm and thus Riemann integrable in the operator norm. Therefore we can introduce the notion that is fundamental in what follows.

[3] A linear operator the inverse of which is bounded is necessarily closed. Thus it is reasonable to require closedness in the definition of the resolvent set. On the other hand closedness of S and the existence of the resolvent imply the boundedness of the latter (homeomorphy theorem).

Definition 2.12. *Let $S : D(S) \to \mathcal{H}$ be a closed operator. If there is a path $\Gamma \subset \rho(S)$ then we call*

$$P = \frac{1}{2\pi i} \int_\Gamma (z\mathbb{1} - S)^{-1} \, dz \tag{2.7}$$

Riesz projection of S with respect to Γ. The part of $\sigma(S)$ enclosed by Γ is called spectral set of S with respect to Γ.

If the spectral set of S with respect to Γ is empty we have $P = 0$. If it coincides with the whole of σ we have $P = \mathbb{1}$. We note the most important properties of Riesz projections.

Theorem 2.13. *Let $S : D(S) \to \mathcal{H}$ be closed and P be the Riesz projection of S with respect to $\Gamma \subset \rho(S)$ according to Definition 2.12. Then the following statements hold:*

1. *$P : \mathcal{H} \to \mathcal{H}$ is bounded.*
2. *$P\varphi \in D(S)$ for all $\varphi \in \mathcal{H}$.*
3. *$SP\varphi = PS\varphi$ for all $\varphi \in D(S)$.*
4. *$SP : \mathcal{H} \to \mathcal{H}$ is bounded.*
5. *The Hilbert space can be decomposed into a direct sum: $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with $\mathcal{H}_1 := \operatorname{ran} P$ and $\mathcal{H}_2 = \operatorname{ran}(\mathbb{1} - P)$.*
6. *Let $\sigma_1(S)$ be the spectral set of S with respect to Γ and $\sigma_2(S) := \sigma(S) \setminus \sigma_1(S)$. The restrictions $PSP : \operatorname{ran} P \to \operatorname{ran} P$ and $(\mathbb{1} - P)S(\mathbb{1} - P) : \operatorname{ran}(\mathbb{1} - P) \to \operatorname{ran}(\mathbb{1} - P)$ have the respective spectra $\sigma_1(S)$ and $\sigma_2(S)$. Moreover the restriction of the resolvent equals the resolvent of the restriction.*

Proof. [10], Section III.4.6. □

Corollary 2.14. *Let $S : D(S) \to \mathcal{H}$ be closed and P the Riesz projection of S with respect to path $\Gamma \subset \rho(S)$. If $\dim \operatorname{ran} P = N < \infty$ then the spectral set of S with respect to Γ contains exactly N eigenvalues (up to multiplicity) and no further spectral values.*

Proof. Since $\operatorname{ran} P$ is an N-dimensional vector space the operator $PSP : \operatorname{ran} P \to \operatorname{ran} P$ has exactly N eigenvalues (up to multiplicity). By Theorem 2.13 the spectrum of $PSP : \operatorname{ran} P \to \operatorname{ran} P$ coincides with the spectral set of S with respect to Γ. □

Theorem 2.15. *In addition to the assumptions of Theorem 2.13 let S be self-adjoint. Then the corresponding Riesz projection is also self-adjoint, i.e., an orthogonal projection.*

Proof. [10], Section V.3.5. □

Theorem 2.11 states that Riesz projections behave continuously under perturbations. We do not want to make this precise at this point but refer to Section 5. Under certain assumptions Riesz projections depend differentiable on perturbations. We only need a special case.

Theorem 2.16. *Let $S_0 : D(S_0) \to \mathcal{H}$ be closed and $S_1 : \mathcal{H} \to \mathcal{H}$ be bounded. We put $S(\beta) := S_0 + \beta S_1$, $\beta \in \mathbb{R}$. If $S_0 = S(0)$ has a Riesz projection $P(0)$ with respect to the path $\Gamma \subset \rho(S_0)$ then there is a $\beta_0 > 0$ such that $\Gamma \subset \rho(S(\beta))$ for all $\beta \in]-\beta_0, \beta_0[$ and the corresponding Riesz projections $P(\beta)$ are continuously differentiable with respect to β on $]-\beta_0, \beta_0[$, i.e. the limit $P'(\beta) = \lim\limits_{h\to 0} \frac{1}{h}(P(\beta+h) - P(\beta))$ exists in the operator norm and depends continuously in the operator norm on β. We have:*

$$P'(\beta) = \frac{1}{2\pi i}\int_\Gamma (z\mathbb{1} - S(\beta))^{-1} S_1 (z\mathbb{1} - S(\beta))^{-1}\, dz. \tag{2.8}$$

Proof. Pick $z \in \rho(S_0)$. Then:

$$z\mathbb{1} - S(\beta) = z\mathbb{1} - (S_0 + \beta S_1) = (z\mathbb{1} - S_0)(\mathbb{1} - \beta(z\mathbb{1} - S_0)^{-1}S_1).$$

By Theorem 2.11 the function $z \mapsto \|(z\mathbb{1} - S_0)^{-1}\|$ is continuous on $\rho(S_0)$ and therefore takes its maximum on the compact set $\Gamma \subset \rho(S_0)$:

$$m := \max_{z\in\Gamma} \|(z\mathbb{1} - S_0)^{-1}\| < \infty.$$

Using this we can estimate:

$$\|\beta(z\mathbb{1} - S_0)^{-1}S_1\| \le |\beta| m \|S_1\|.$$

We choose $\beta_0 > 0$ such that $\beta_0 m\|S_1\| < 1$. Then the Neumann series gives the existence and boundedness of $(z\mathbb{1} - S(\beta))^{-1}$ for all $\beta \in]-\beta_0, \beta_0[$ and all $z \in \Gamma$. We have:

$$\|(z\mathbb{1} - S(\beta))^{-1}\| \le \frac{m}{1 - \beta_0 m\|S_1\|}.$$

We show that the difference quotient converges to the expression in (2.8). Let $z \in \Gamma$, $\beta \in]-\beta_0, \beta_0[$, and h be sufficiently small:

$$\begin{aligned}
&(z\mathbb{1} - S(\beta))^{-1}S_1(z\mathbb{1} - S(\beta))^{-1} - \frac{1}{h}\left[(z\mathbb{1} - S(\beta+h))^{-1} - (z\mathbb{1} - S(\beta))^{-1}\right]\\
&= (z\mathbb{1} - S(\beta))^{-1}S_1(z\mathbb{1} - S(\beta))^{-1} - (z\mathbb{1} - S(\beta+h))^{-1}S_1(z\mathbb{1} - S(\beta))^{-1}\\
&= \left[(z\mathbb{1} - S(\beta))^{-1} - (z\mathbb{1} - S(\beta+h))^{-1}\right] S_1(z\mathbb{1} - S(\beta))^{-1}\\
&= -h(z\mathbb{1} - S(\beta))^{-1}S_1(z\mathbb{1} - S(\beta+h))^{-1}S_1(z\mathbb{1} - S(\beta))^{-1}.
\end{aligned}$$

From this we obtain an estimate for $\beta \in]-\beta_0, \beta_0[$:

$$\begin{aligned}
\Big\| \int_\Gamma (z\mathbb{1} - S(\beta))^{-1}S_1(z\mathbb{1} - S(\beta))^{-1}\,dz - \frac{1}{h}\int_\Gamma (z\mathbb{1} - S(\beta+h))^{-1} - (z\mathbb{1} - S(\beta))^{-1}\,dz\Big\| \\
\le |\Gamma| m^2 \|S_1\|^2 \frac{m}{1 - h_0 m\|S_1\|}|h|.
\end{aligned}$$

This implies differentiability and the formula (2.8).

We show with the aid of (2.8) that $P'(\beta)$ is continuous with respect to β on $]-\beta_0, \beta_0[$. To this end we use a similar formula as above ($\beta_1, \beta_2 \in]-\beta_0, \beta_0[$):

$$\begin{aligned}
&(z\mathbb{1} - S(\beta_2))^{-1}S_1(z\mathbb{1} - S(\beta_2))^{-1} - (z\mathbb{1} - S(\beta_1))^{-1}S_1(z\mathbb{1} - S(\beta_1))^{-1}\\
&\quad = (\beta_1 - \beta_2)(z\mathbb{1} - S(\beta_2))^{-1}S_1\left[(z\mathbb{1} - S(\beta_2))^{-1} + (z\mathbb{1} - S(\beta_1))^{-1}\right] S_1(z\mathbb{1} - S(\beta_1))^{-1}.
\end{aligned}$$

From this we obtain the estimate:

$$\|(z\mathbb{1} - S(\beta_2))^{-1}S_1(z\mathbb{1} - S(\beta_2))^{-1} - (z\mathbb{1} - S(\beta_1))^{-1}S_1(z\mathbb{1} - S(\beta_1))^{-1}\| \\ \leq 2|\beta_1 - \beta_2| \left(\frac{m}{1 - \beta_0 m\|S_1\|}\right)^3 \|S_1\|^2.$$

This can be used to show the continuity of $P'(\beta)$. □

With the aid of Theorem 2.16 we can examine the case that S_0 and S_1 commute.

Corollary 2.17. *Let $S_0 : D(S_0) \to \mathcal{H}$ be closed and $S_1 : \mathcal{H} \to \mathcal{H}$ be bounded. For all $\varphi \in \mathcal{H}$ let $S_1\varphi \in D(S_0)$ and $S_0S_1\varphi = S_1S_0\varphi$. If P is the Riesz projection of S_0 with respect to the path $\Gamma \subset \rho(S_0)$ then for $\beta \in \mathbb{R}$ sufficiently small P is also the Riesz projection of $S(\beta) := S_0 + \beta S_1$ with respect to Γ.*

Proof. As in Theorem 2.16 it is seen that $\Gamma \subset \rho(S(\beta))$ for β sufficiently small. Let $P(\beta)$ be the corresponding Riesz projections. We have:

$$\begin{aligned} P'(\beta) &= \frac{1}{2\pi i}\int_\Gamma (z\mathbb{1} - S(\beta))^{-1}S_1(z\mathbb{1} - S(\beta))^{-1}\,dz \\ &= S_1\frac{1}{2\pi i}\int_\Gamma [(z\mathbb{1} - S(\beta))^{-1}]^2\,dz \\ &= \frac{1}{2\pi i}\int_\Gamma [(z\mathbb{1} - S(\beta))^{-1}]^2\,dzS_1. \end{aligned}$$

Here we have used that S_1 commutes with $S(\beta)$. On the other hand $P(\beta) = P(\beta)^2$ implies:

$$P'(\beta) = P(\beta)P'(\beta) + P'(\beta)P(\beta).$$

We show $P(\beta)P'(\beta) = 0$. We have

$$P(\beta)P'(\beta) = \frac{1}{2\pi i}\int_\Gamma P(\beta)[(z\mathbb{1} - S(\beta))^{-1}]^2\,dzS_1 = \frac{1}{2\pi i}\int_\Gamma [P(\beta)(z\mathbb{1} - S(\beta))^{-1}]^2\,dzS_1,$$

since $P(\beta)$ being a Riesz projection commutes with $(z\mathbb{1} - S(\beta))^{-1}$. All singularities of $P(\beta)(z\mathbb{1} - S(\beta))^{-1}$ are enclosed by Γ by Theorem 2.13 and thus are a bounded set. Hence we have:

$$\|P(\beta)(z\mathbb{1} - S(\beta))^{-1}\| \leq \frac{C}{|z|}$$

for sufficiently large z implying together with the standard estimate for contour integrals:

$$\|\int_\Gamma [P(\beta)(z\mathbb{1}-S(\beta))^{-1}]^2\,dzS_1\| = \|\int_{K_r} [P(\beta)(z\mathbb{1}-S(\beta))^{-1}]^2\,dzS_1\| \leq \frac{C}{r^2}r\|S_1\| = \frac{C}{r}\|S_1\| \to 0$$

for $r \to \infty$. Analogously we see $P'(\beta)P(\beta) = 0$ whence $P'(\beta) = 0$ follows. This shows $P(\beta) = P(0) = P$. □

Integrals of the type arising from the differentiation of Riesz projections bear their own name.

Definition 2.18. *Let $S : D(S) \to \mathcal{H}$ be closed and $\tilde{S} : \mathcal{H} \to \mathcal{H}$ be bounded. Let further $\Gamma \subset \rho(S)$ be an integration path. Then*

$$\Phi(\tilde{S}) := \frac{1}{2\pi i} \int_\Gamma (z\mathbb{1} - S)^{-1} \tilde{S} (z\mathbb{1} - S)^{-1} \, dz \tag{2.9}$$

is called Friedrichs integral.[4] *The integral is defined as a Riemann integral in the operator norm.*

Friedrichs integrals can be used to solve commutators.

Lemma 2.19. *Let $S : D(S) \to \mathcal{H}$ be closed and $\tilde{S} : \mathcal{H} \to \mathcal{H}$ be bounded. If $\Phi(\tilde{S})$ is the Friedrichs integral of $\tilde{S}$ with respect to the integration path $\Gamma \subset (\rho(S))$ according to Definition 2.18 then:*

1. *$\Phi(\tilde{S}) : \mathcal{H} \to \mathcal{H}$ is bounded.*
2. *$\Phi(\tilde{S})\varphi \in D(S)$ for all $\varphi \in \mathcal{H}$ and $S\Phi(\tilde{S}) : \mathcal{H} \to \mathcal{H}$ is bounded.*
3. *If P is the Riesz projection of S with respect to Γ then for all $\varphi \in D(S)$:*

$$[\Phi(\tilde{S}), S]\varphi = [\tilde{S}, P]\varphi.$$

Proof. Since the resolvent depends continuously in the operator norm on z the Riemann sums

$$\Phi_n := \frac{1}{2\pi i} \sum_{j=1}^{n} (z_j - z_{j-1})(\tilde{z}_j \mathbb{1} - S)^{-1} \tilde{S} (\tilde{z}_j \mathbb{1} - S)^{-1}$$

converge in the operator norm to the Friedrichs integral.

1. Since the resolvent and $\tilde{S}$ are bounded so is $\Phi(\tilde{S})$ being the limit of the Riemann sums.

2. $S(z\mathbb{1} - S)^{-1} = z(z\mathbb{1} - S)^{-1} - z\mathbb{1}$ implies that $S\Phi_n$ is bounded and converges in the operator norm. Thus we know for every $\varphi \in \mathcal{H}$ that $\Phi_n\varphi$ and $S\Phi_n\varphi$ converge. Since S is closed we conclude $\Phi(\tilde{S})\varphi \in D(S)$ for all $\varphi \in \mathcal{H}$ and that $S\Phi(\tilde{S})$ is bounded.

3. For $\varphi \in D(S)$ we calculate:

$$\begin{aligned}
[\Phi(\tilde{S}), S]\varphi &= \frac{1}{2\pi i} \int_\Gamma [(z\mathbb{1} - S)^{-1} \tilde{S} (z\mathbb{1} - S)^{-1}, S]\varphi \, dz \\
&= \frac{1}{2\pi i} \int_\Gamma [(z\mathbb{1} - S)^{-1} \tilde{S} (z\mathbb{1} - S)^{-1}, -z\mathbb{1} + S]\varphi \, dz \\
&= \frac{1}{2\pi i} \int_\Gamma (-(z\mathbb{1} - S)^{-1} \tilde{S} + \tilde{S} (z\mathbb{1} - S)^{-1})\varphi \, dz \\
&= [\tilde{S}, \frac{1}{2\pi i} \int_\Gamma (z\mathbb{1} - S)^{-1} \varphi \, dz] \\
&= [\tilde{S}, P]\varphi.
\end{aligned}$$

The last step is just the definition of Riesz projections as in (2.7). □

[4] The map $\tilde{S} \mapsto \Phi(\tilde{S})$ is sometimes called Friedrichs' Gamma operation. In the literature often unitary operators are used instead of the resolvent. See [10], X.5

2.4 Groups of bounded operators

The following presentation essentially follows [11] and [10].

Definition 2.20. *A one-parameter family $U(t) : \mathcal{H} \to \mathcal{H}$, $t \in \mathbb{R}$, of bounded linear operators is called strongly continuous group of bounded linear operators if the following holds:*

1. $U(0) = \mathbb{1}$.
2. $U(t_1 + t_2) = U(t_1)U(t_2)$, $t_1, t_2 \in \mathbb{R}$.
3. *The map $U(\cdot)\varphi : t \mapsto U(t)\varphi$ is continuous in the Hilbert space norm for all $\varphi \in \mathcal{H}$.*

The linear operator $S : D(S) \to \mathcal{H}$ defined by

$$D(S) := \left\{ \varphi \in \mathcal{H} | \lim_{t \to 0} \frac{1}{t}(U(t)\varphi - \varphi) \text{ exists} \right\}, \quad S\varphi := \lim_{t \to 0} \frac{1}{t}(U(t)\varphi - \varphi), \ \varphi \in D(S)$$

is called generator of the group.

In what follows we shall say instead of strongly continuous groups of bounded linear operators simply group of operators or more simple just group if there is no misunderstanding. The generator S is also said to generate the corresponding group. The generator of a group of operators is uniquely determined. The converse is also true, i.e., an operator can generate at most one group.

Theorem 2.21. *Let S_1 and S_2 be the generators of the groups $U_1(t)$ and $U_2(t)$, respectively. Then $S_1 = S_2$ implies $U_1(t) = U_2(t)$ for all $t \in \mathbb{R}$.*

Proof. Follows from [11], Th. 1.2.6. □

By Theorem 2.21 we may talk about the group belonging to S. We have the following fundamental existence theorem.

Theorem 2.22. *Let $M \geq 1$ and $\omega \geq 0$. An operator $S : D(S) \to \mathcal{H}$ is the generator of a group $U(t)$ with*

$$\|U(t)\| \leq M e^{\omega|t|}, \ t \in \mathbb{R},$$

if and only if the following holds:

1. *S is densely defined and closed.*
2. *$\lambda \in \mathbb{R}$ and $|\lambda| > \omega$ implie $\lambda \in \rho(S)$. For all $n \in \mathbb{N}$ there holds the estimate:*

$$\| \left[(\lambda \mathbb{1} - S)^{-1} \right]^n \| \leq M(|\lambda| - \omega)^{-n}.$$

Proof. [11], Th. 1.6.3. □

We relate groups of operators with differential equations.

Theorem 2.23. *Let $U(t)$ be a group of operators with generator S. Then $U(t)\varphi \in D(S)$, $t \in \mathbb{R}$, for all $\varphi \in D(S)$ and the function $t \mapsto U(t)\varphi$ is continuously differentiable with:*

$$\frac{d}{dt}U(t)\varphi = SU(t)\varphi = U(t)S\varphi. \tag{2.10}$$

Proof. Follows from [11], Th. 1.2.4. □

Theorems 2.22 and 2.23 have an important consequence.

Corollary 2.24. *Let $A : D(A) \to \mathcal{H}$ be self-adjoint and $U(t)$ be the group generated by $-iA$. Then $U(t)$ is unitary for all $t \in \mathbb{R}$.*

Proof. We check the conditions of Theorem 2.22. $-iA$ is densely defined and closed since A as a self-adjoint operator is. The inequality for the resolvent in Theorem 3.1 yields for $\lambda \in \mathbb{R} \setminus \{0\}$:

$$\| \left[(\lambda \mathbb{1} + iA)^{-1}\right]^n \| \leq \|(i\lambda\mathbb{1} - A)^{-1}\|^n \leq |\lambda|^{-n}.$$

Hence $-iA$ generates a group $U(t)$ with $M = 1$ and $\omega = 0$ in Theorem 2.22.

In order to prove the unitarity we differentiate $\|U(t)\varphi\|^2$ and notice the differential equation (2.10):

$$\frac{d}{dt}\|U(t)\varphi\|^2 = 2\,\mathrm{Re}(U(t)\varphi, U'(t)\varphi) = -2\,\mathrm{Re}\, i(U(t)\varphi, AU(t)\varphi) = 0.$$

We conclude that $U(t)$ is isometric. Combining this with the group property implies the statement. □

By Theorem 2.23 the group $U(t)$ can be used to obtain solutions of differential equations. Actually these are all solutions.

Theorem 2.25. *Let $S : D(S) \to \mathcal{H}$ be densely defined with $\rho(S) \neq \emptyset$. Then the initial value problem*

$$\frac{d\varphi(t)}{dt} = S\varphi(t),\ t \in \mathbb{R},\ \varphi(0) = \psi, \tag{2.11}$$

has for every initial value $\psi \in D(S)$ a unique continuously differentiable solution $\varphi(t) \in \mathcal{H}$ if and only if S is the generator of a group $U(t)$ of operators according to Definition 2.20. The solution is given by $\varphi(t) = U(t)\psi$.

Proof. [11], Th. 4.1.3. □

The inhomogeneous initial value problem can also be solved with the aid of the group $U(t)$.

Theorem 2.26. *Let $S : D(S) \to \mathcal{H}$ be the generator of a group $U(t)$ of operators and let $f(\cdot) : t \mapsto f(t) \in \mathcal{H}$ be continuously differentiable. Then the function*

$$g(t) := \int_{t_0}^{t} U(t-\tau)f(\tau)\,d\tau$$

is also continuously differentiable. We have $g(t) \in D(S)$ and

$$\frac{d}{dt}g(t) = f(t) + Sg(t),\ g(t_0) = 0.$$

Proof. Follows from [10], IX.1.5. □

The group generated by an operator can be written down explicitly. Bounded operators make no problems concerning domains of definition.

Theorem 2.27. *Let $S : \mathcal{H} \to \mathcal{H}$ be bounded and $\Gamma \subset \rho(S)$ be an integration path enclosing the spectrum of S. Then the group $U(t)$ generated by S is given by the Dunford-Taylor integral*

$$U(t) = \frac{1}{2\pi i} \int_\Gamma e^{zt} (z\mathbb{1} - S)^{-1} \, dz. \tag{2.12}$$

The integral is to be understood as Riemann integral in the operator norm. A further representation is given by a power series:

$$U(t) = \mathbb{1} + \sum_{j=1}^{\infty} \frac{1}{j!} t^j S^j \tag{2.13}$$

the series cconverging in the operator norm. In particular $U(t)$ is differentiable with respect to t in the operator norm.

Proof. [10], Sections IX.1.6 und IX.1.2. □

The Dunford-Taylor integral (2.12) can be shown to exist under weaker conditions if the integration path is appropriately chosen (see [10], IX.1.6). The application in Section 5 requires a variant.

Corollary 2.28. *Let $S : \mathcal{H} \to \mathcal{H}$ be bounded and $\Gamma \subset \rho(S)$ be an integration path enclosing the spectrum of S. Then the group $U(t)$ generated by $-iS$ is given by the Dunford-Taylor integral*

$$U(t) = \frac{1}{2\pi i} \int_\Gamma e^{-izt} (z\mathbb{1} - S)^{-1} \, dz. \tag{2.14}$$

The integral is to be understood as Riemann integral in the operator norm.

Proof. Substitute $z \to -iz$ in (2.12). □

The formulas (2.12) or (2.14) permit an important estimate.

Corollary 2.29. *Let $S : \mathcal{H} \to \mathcal{H}$ be bounded and $U(t)$ be the group generated by $-iS$. If*

$$\inf_{\mu \in \sigma(S)} \operatorname{Im} \mu > 0$$

then there are constants $M \geq 0$ und $\delta > 0$ such that for all $t \leq 0$:

$$\|U(t)\| \leq M e^{\delta t}. \tag{2.15}$$

Proof. In formula (2.14) we can choose Γ in such a way that $\delta := \inf_{z \in \Gamma} \operatorname{Im} z > 0$. With the aid of the standard estimate for contour integrals we obtain for $t \leq 0$:

$$\|U(t)\| \leq \frac{|\Gamma|}{2\pi} \sup_{z \in \Gamma} e^{t \operatorname{Im} z} \|(z\mathbb{1} - S)^{-1}\| \leq \frac{|\Gamma|}{2\pi} e^{t\delta} \sup_{z \in \Gamma} \|(z\mathbb{1} - S)^{-1}\|.$$

This proves the statement. □

Theorem 2.27 has a converse, i.e., the resolvent can be expressed by the group.

Theorem 2.30. *Let $S : D(S) \to \mathcal{H}$ be the generator of a group $U(t)$ and $K : \mathcal{H} \to \mathcal{H}$ be a bounded operator such that*

$$\|U(t)K\| \le Me^{\omega t}$$

for all $t \ge 0$ with constants $M \ge 0$ and $\omega \in \mathbb{R}$.[5] *Then for all z, $\operatorname{Re} z > \omega$, and for all $\varphi \in \mathcal{H}$:*

$$(z\mathbb{1} - S)^{-1}K\varphi = \int_{-\infty}^{0} e^{zt}U(-t)K\varphi\, dt. \tag{2.16}$$

Proof. Follows from the proofs of [11], Th. 1.5.3. and Th. 1.3.1. □

The substitution $t \to -t$ brings (2.16) into the well known form

$$(z\mathbb{1} - S)^{-1}\varphi = \int_{0}^{\infty} e^{-zt}U(t)\varphi\, dt.$$

We use this unusual from in order to keep the connection of the integral formulas for groups of operators and for Riesz projections via the adiabatic theorem (see the introduction).

We examine how a group of operators behaves under the perturbation of the generator. An application of the existence theorem 2.22 leads to the following stability result.

Theorem 2.31. *Let $S_0 : D(S_0) \to \mathcal{H}$ be the generator of a group $U_0(t)$ with*

$$\|U_0(t)\| \le Me^{\omega|t|},$$

$M \ge 1$ and $\omega \ge 0$. Let further $S_1 : \mathcal{H} \to \mathcal{H}$ be bounded. Then $S_0 + S_1 : D(S_0) \to \mathcal{H}$ generates also a group $U(t)$ having the property:

$$\|U(t)\| \le Me^{(\omega + M\|S_1\|)|t|}.$$

Proof. [9], Th. 13.2.2. □

In order to exactly study the connection between the groups $U_0(t)$ and $U(t)$ from Lemma 2.31 we derive an integral equation.

Theorem 2.32. *Let $S_0 : D(S_0) \to \mathcal{H}$ be the generator of a group $U_0(t)$ and $S_1 : \mathcal{H} \to \mathcal{H}$ be bounded. If $U(t)$ is the group generated by $S_0 + S_1$ then for all $\varphi \in \mathcal{H}$ we have:*

$$(U_0(-t_1)U(t_1) - U_0(-t_0)U(t_0))\varphi = \int_{t_0}^{t_1} U_0(-\tau)S_1U(\tau)\varphi\, d\tau, \tag{2.17}$$

$$U(t)\varphi = U_0(t)\varphi + \int_{0}^{t} U_0(t-\tau)S_1U(\tau)\varphi\, d\tau. \tag{2.18}$$

If in addition for all $\varphi \in \mathcal{H}$:

$$\int_{-\infty}^{t_1} \|U_0(-\tau)S_1U(\tau)\varphi\|\, d\tau < \infty, \quad \lim_{t\to-\infty} \|U_0(-t)U(t)\varphi\| = 0 \tag{2.19}$$

then for all $\varphi \in \mathcal{H}$:

$$U_0(-t_1)U(t_1)\varphi = \int_{-\infty}^{t_1} U_0(-\tau)S_1U(\tau)\varphi\, d\tau. \tag{2.20}$$

[5] Because of the operator K the constant ω may be negative contrary to the usual estimate for $U(t)$ what will become important later.

Proof. See [11], Section 3.1. The function $t \mapsto U_0(-t)U(t)\varphi$ is differentiable for $\varphi \in D(S_0)$ and we have:

$$\begin{aligned}\frac{d}{dt}U_0(-t)U(t)\varphi &= \frac{d}{dt}U_0(-t)\cdot U(t)\varphi + U_0(-t)\frac{d}{dt}U(t)\varphi \\ &= -U_0(-t)S_0U(t)\varphi + U_0(-t)(S_0+S_1)U(t)\varphi \\ &= U_0(-t)S_1U(t)\varphi.\end{aligned}$$

Integrating gives the formula (2.17) for $\varphi \in D(S_0)$. Since $U_0(t)$, $U(t)$ and S_1 are bounded the formula is valid for all $\varphi \in \mathcal{H}$. The integral equation (2.18) follows by setting $t_0 = 0$ and $t_1 = t$.

The condition (2.19) implies (2.20) via the formula (2.18). □

The group $U(t)$ is uniquely determined by the integral equation.

Theorem 2.33. *The integral equation (2.18) has exactly one strongly continuous solution $\tilde{U}(t)$. It is given by:*

$$\tilde{U}(t) = \sum_{j=0}^{\infty} U_j(t),\ U_{j+1}(t)\varphi = \int_0^t U_0(t-\tau)S_1U_j(\tau)\varphi\, d\tau,\ j \geq 0,\ \varphi \in \mathcal{H}. \tag{2.21}$$

The series converges in the operator norm locally uniformly in t.

Proof. [10], IX.2.1. or [11], Prop. 3.1.2. □

From the integral equation we deduce that the group depends continuously on perturbations.

Theorem 2.34. *Let $S_0 : D(S_0) \to \mathcal{H}$ be closed and the generator of a group $U_0(t)$ with:*

$$\|U_0(t)\| \leq Me^{\omega|t|}$$

with constants $M \geq 1$, $\omega \geq 0$. If $S_1 : \mathcal{H} \to \mathcal{H}$ is bounded then $S_0 + S_1$ also generates a group $U(t)$ and we have for all $t \in \mathbb{R}$:

$$\|U(t) - U_0(t)\| \leq Me^{\omega|t|}(e^{M\|S_1\||t|} - 1).$$

Proof. [11], Corollary 3.1.3. □

Under certain conditions the group depends even differentiably on the generator. For our purposes it is enough to consider a special case.

Theorem 2.35. *Let $S_0 : D(S_0) \to \mathcal{H}$ be closed and the generator of a group $U_0(t)$ with*

$$\|U_0(t)\| \leq Me^{\omega|t|}$$

with constants $M \geq 1$, $\omega \geq 0$. If $S_1 : \mathcal{H} \to \mathcal{H}$ is bounded then $S_0 + \beta S_1$, $\beta \in \mathbb{R}$, generates a group $U(t, \beta)$ depending continuously differentiably in the operator norm on β for every fixed $t \in \mathbb{R}$. For all $\varphi \in \mathcal{H}$ we have:

$$(\frac{\partial}{\partial\beta}U(t,\beta))\varphi = U(t,\beta)\int_0^t U(-\tau,\beta)S_1U(\tau,\beta)\varphi\, d\tau = \int_0^t U(t-\tau,\beta)S_1U(\tau,\beta)\varphi\, d\tau. \tag{2.22}$$

Proof. We consider the difference quotient $\frac{1}{h}(U(t,\beta+h)-U(t,\beta))$. By Theorem 2.32 we have the following integral equation:

$$U(t,\beta+h)\varphi = U(t,\beta)\varphi + h\int_0^t U(t-\tau,\beta)S_1U(\tau,\beta+h)\varphi\,d\tau,\ \varphi\in\mathcal{H}. \tag{2.23}$$

Iterating (2.23) and rearranging for the difference quotient gives:

$$\begin{aligned}&\frac{1}{h}(U(t,\beta+h)\varphi-U(t,\beta)\varphi)=\\&\int_0^t U(t-\tau,\beta)S_1U(\tau,\beta)\varphi\,d\tau + h\int_0^t U(t-\tau,\beta)S_1\int_0^\tau U(\tau-\tau',\beta)S_1U(\tau',\beta+h)\varphi\,d\tau'\,d\tau.\end{aligned} \tag{2.24}$$

From Theorem 2.31 we have the estimate:

$$\|U(t,\beta)\|\le Me^{(\omega+\beta M\|S_1\|)|t|}. \tag{2.25}$$

Then (2.24) shows that the difference quotient converges in the operator norm and that formula (2.22) is correct.

We show that $\frac{\partial}{\partial\beta}U(t,\beta)$ is continuous in β in the operator norm for fixed t. (2.22) implies:

$$\begin{aligned}\frac{\partial}{\partial\beta}U(t,\beta+h)\varphi-\frac{\partial}{\partial\beta}U(t,\beta)\varphi = \int_0^t &[U(t-\tau,\beta+h)S_1(U(\tau,\beta+h)-U(\tau,\beta))\\ &+(U(t-\tau,\beta+h)-U(t-\tau,\beta))S_1U(\tau,\beta)]\,\varphi\,d\tau.\end{aligned} \tag{2.26}$$

Theorem 2.34 yields the estimate:

$$\|U(t,\beta+h)-U(t,\beta)\|\le Me^{(\omega+\beta M\|S_1\|)}\left(e^{|h|M\|S_1\||t|}-1\right).$$

Applying this together with (2.25) to (2.26) completes the proof. □

We consider the case that the operators S_0 and S_1 commute.

Theorem 2.36. *Let $S_0: D(S_0)\to\mathcal{H}$ be the generator of a group $U_0(t)$ and $S_1:\mathcal{H}\to\mathcal{H}$ be bounded. For all $\varphi\in D(S_0)$ let $S_1\varphi\in D(S_0)$ and $S_1S_0\varphi=S_0S_1\varphi$ for all $\varphi\in D(S_0)$. Then $U_0(t)$ and S_1 commute, i.e. for all $\varphi\in\mathcal{H}$:*

$$U_0(t)S_1\varphi = S_1U_0(t)\varphi.$$

Proof. By Theorem 2.25 the unique solution to

$$\varphi'(t)=S_0\varphi(t),\ \varphi(0)=\psi\in D(S_0) \tag{2.27}$$

is given by

$$\varphi(t)=U_0(t)\psi.$$

We multiply (2.27) by S_1. Since S_1 is bounded we have $(S_1\varphi')(t)=(S_1\varphi)'(t)$. It follows:

$$(S_1\varphi)'(t)=S_1S_0\varphi(t)=S_0S_1\varphi(t),\ S_1\varphi(0)=S_1\psi.$$

With $\tilde{\varphi}(t) := S_1\varphi(t)$ we have:

$$\tilde{\varphi}'(t) = S_0\tilde{\varphi}(t), \ \tilde{\varphi}(0) = S_1\psi. \tag{2.28}$$

By Theorem 2.25 the unique solution to (2.28) is given by:

$$\tilde{\varphi}(t) = U_0(t)S_1\psi.$$

Thus we obtain for all $\psi \in D(S_0)$:

$$U_0(t)S_1\psi = S_1U_0(t)\psi.$$

Since $U_0(t)$ and S_1 are bounded this is valid for for all $\psi \in \mathcal{H}$. □

The group generated by S_1 can be split off.

Theorem 2.37. *Let S_0, S_1 and $U_0(t)$ be as in Theorem 2.36. Let $U(t)$ and $U_1(t)$ the groups generated by $S_0 + S_1$ and S_1, respectively. Then:*

$$U(t) = U_0(t)U_1(t) = U_1(t)U_0(t).$$

Proof. By Theorem 2.32 $U(t)$ satisfies the integral equation:

$$U(t)\varphi = U_0(t)\varphi + \int_0^t U_0(t-\tau)S_1U(\tau)\varphi\,d\tau$$

for all $\varphi \in \mathcal{H}$. We show that U_0U_1 is also a solution to the integral equation. We notice $S_1U_0(\tau) = U_0(\tau)S_1$ (Theorem 2.36) and the group property. For $\varphi \in \mathcal{H}$ we have:

$$\begin{aligned}
\int_0^t U_0(t-\tau)S_1U_0(\tau)U_1(\tau)\varphi\,d\tau &= \int_0^t U_0(t-\tau)U_0(\tau)S_1U_1(\tau)\varphi\,d\tau \\
&= \int_0^t U_0(t)S_1U_1(\tau)\varphi\,d\tau \\
&= U_0(t)\int_0^t \frac{d}{d\tau}U_1(\tau)\varphi\,d\tau \\
&= U_0(t)(U_1(t) - \mathbb{1})\varphi \\
&= (U(t) - U_0(t))\varphi.
\end{aligned}$$

Here we have used the differential equation for $U_1(t)$ according to Theorem 2.23. Since S_1 is bounded we did not care about domains of definition. From unique solvability of the integral equation we conclude $U = U_0U_1$.

We have to show $U_0U_1 = U_1U_0$. Since S_1 is bounded we can write U_1 as power series:

$$U_1(t) = \sum_{j=0}^{\infty} \frac{1}{j!}t^j S_1^j.$$

By Theorem 2.36 we have $U_0S_1 = S_1U_0$. From this we see easily $U_0S_1^j = S_1^jU_0$. This yields:

$$U_0(t)U_1(t) = U_0(t)\sum_{j=0}^{\infty}\frac{1}{j!}t^j S_1^j = \sum_{j=1}^{\infty}\frac{1}{j!}t^j U_0(t)S_1^j = \sum_{j=1}^{\infty}\frac{1}{j!}t^j S_1^j U_0(t) = U_1(t)U_0(t).$$

This completes the proof. □

A projection can be separated from the group under certain conditions.

Lemma 2.38. *Let $S : D(S) \to \mathcal{H}$ be the generator of a group $U(t)$ and $P : \mathcal{H} \to \mathcal{H}$ a bounded projection having the property $P\varphi \in D(S)$ for $\varphi \in D(S)$. Let P commute with $U(t)$. If $\tilde{U}(t)$ is the group generated by SP then $\tilde{U}(t)P = U(t)P$.*

Proof. We show that $\tilde{U}(t)P$ and $U(t)P$ solve the same initial value problem. Because of $\tilde{U}(0) = \mathbb{1}$ und $U(0) = \mathbb{1}$ the initial values coincide. $\varphi \in D(S)$ implies $P\varphi \in D(S)$ and we have

$$i\frac{d}{dt}\tilde{U}(t)P\varphi = SP\tilde{U}(t)P\varphi = S\tilde{U}(t)P^2\varphi = S\tilde{U}(t)P\varphi.$$

On the other hand we have

$$i\frac{d}{dt}U(t)P\varphi = SU(t)P\varphi.$$

The uniqueness of the solution to the initial value problem (Theorem 2.36) now yields the statement. □

We compute the group of a concrete generator, which will appear in Section 5.

Lemma 2.39. *Let $P : \mathcal{H} \to \mathcal{H}$ be a bounded projection. We put $J = 2P - \mathbb{1}$. Then the group $U(t)$ generated by γJ, $\gamma \in \mathbb{C}$, is given by:*

$$U(t)P = PU(t) = e^{\gamma t}P,\ U(t)(\mathbb{1} - P) = (\mathbb{1} - P)U(t) = e^{-\gamma t}(\mathbb{1} - P).$$

Proof. J is bounded and commutes with P. We have $J^2 = 4P^2 - 4P + \mathbb{1} = \mathbb{1}$. We employ the power series for $U(t)$:

$$U(t) = \sum_{j=1}^{\infty} \frac{1}{j!}(\gamma t)^j J^j = \sum_{k=0}^{\infty} \frac{1}{(2k)!}(\gamma t)^{2k}\mathbb{1} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(\gamma t)^{2k+1}J.$$

$JP = (2P - \mathbb{1})P = P$ and $J(\mathbb{1} - P) = (2P - \mathbb{1})(\mathbb{1} - P) = P - \mathbb{1}$ imply the lemma. □

Finally we introduce the Green operators.

Definition 2.40. *Let $U(t)$ be a group of bounded operators with generator $S : D(S) \to \mathcal{H}$, $K : \mathcal{H} \to \mathcal{H}$ be a bounded operator, and $\gamma \in \mathbb{C} \setminus \{0\}$. Then*

$$G(t, t') := \begin{cases} \gamma U(t)KU(-t'), & t \leq t', \\ \gamma U(t)[K - \mathbb{1}]U(-t'), & t > t', \end{cases}$$

is called Green operator with respect to K with generator S to the parameter γ.[6]

G formally satisfies the differential equation:

$$\frac{1}{\gamma}\left(\frac{\partial}{\partial t} - S\right) G(t, t') = \delta(t - t')\mathbb{1}. \tag{2.29}$$

This may be justified in the sense of weak differentiability. Since we do not need this later we will not stress this point. We just note:

[6] In the literature $G(t, t')$ is sometimes called bi-semi group and S accordingly bi-generator. See [3], [4], [5]. The parameter γ appears with view on Sections 4 and 5 where we have a factor i because of the Schrödinger equation.

Theorem 2.41. *Let G be the Green operator according to Definition 2.40. For $\varphi \in D(S)$ let $K\varphi \in D(S)$. Then for all $\varphi \in D(S)$ G satisfies the differential equation:*

$$\frac{\partial}{\partial t} G(t,t')\varphi = SG(t,t')\varphi,\ t \neq t'. \tag{2.30}$$

Proof. By Theorem 2.23 we have $U(-t')\varphi \in D(S)$ for $\varphi \in D(S)$ and by assumption we have $KU(-t')\varphi \in D(S)$. Thus $G(\cdot, t')$ is strongly differentiable. The differential equation then follows immediately from the definition of $G(t,t')$. ☐

Green operators satisfy an integral equation analogous to equation (2.18) for the group $U(t)$.[7]

Theorem 2.42. *Let G_0 be the Green operator according to Definition 2.40 with respect to K_0 to the parameter γ and $S_1 : \mathcal{H} \to \mathcal{H}$ be bounded. Then a Green operator G can be defined with respect to K to the parameter γ. For all $\varphi \in \mathcal{H}$ we have:*

$$\begin{aligned} G(t,t')\varphi + \frac{1}{\gamma} G_0(t,T_0)G(T_0,t')\varphi - \frac{1}{\gamma} G_0(t,T_1)G(T_1,t')\varphi \\ = G_0(t,t')\varphi - \frac{1}{\gamma}\int_{T_0}^{T_1} G_0(t,\tau)S_1G(\tau,t')\varphi\, d\tau. \end{aligned} \tag{2.31}$$

If in addition for $\varphi \in \mathcal{H}$:

$$\int_{-\infty}^{T_1} \|U_0(-\tau)S_1U(\tau)\varphi\|\, dt < \infty,\ \lim_{t\to-\infty} \|U_0(-t)U(t)\varphi\| = 0 \tag{2.32}$$

then for all $\varphi \in \mathcal{H}$:

$$G(t,t')\varphi - \frac{1}{\gamma} G_0(t,T_1)G(T_1,t')\varphi = G_0(t,t')\varphi - \frac{1}{\gamma}\int_{-\infty}^{T_1} G_0(t,\tau)S_1G(\tau,t')\varphi\, d\tau. \tag{2.33}$$

Proof. By Theorem 2.31 S generates a group of operators, too. Thus a Green operator to S can be defined.

In what follows all equations are to be applied to $\varphi \in \mathcal{H}$. Let $t \leq t'$. We split up the integral in (2.31)

$$\int_{T_0}^{T_1} = \int_{T_0}^{t} + \int_{t}^{t'} + \int_{t'}^{T_1}$$

and insert the definitions of G and G_0:

$$\begin{aligned} \frac{1}{\gamma^2}\int_{T_0}^{T_1} G_0(t,\tau)S_1G(\tau,t')\, d\tau = &\int_{T_0}^{t} U_0(t)[K_0 - \mathbb{1}]U_0(-\tau)S_1U(\tau)KU(-t')\, d\tau \\ &+ \int_{t}^{t'} U_0(t)K_0U_0(-\tau)S_1U(\tau)KU(-t')\, d\tau \\ &+ \int_{t'}^{T_1} U_0(t)K_0U_0(-\tau)S_1U(\tau)[K - \mathbb{1}]U(-t')\, d\tau. \end{aligned}$$

[7] The integral equation for $U(t)$ is of Volterra type while here we have a Fredholm type equation. Therefore the upper boundary point is called T_1 and not t.

At this point we can use the integration rule from Theorem 2.32:

$$\begin{aligned}
&\frac{1}{\gamma^2}\int_{T_0}^{T_1} G_0(t,\tau)S_1G(\tau,t')\,d\tau \\
&= U_0(t)[K_0-\mathbb{1}]\left\{U_0(-t)U(t)-U_0(-T_0)U(T_0)\right\}KU(-t') \\
&\quad + U_0(t)K_0\left\{U_0(-t')U(t')-U_0(-t)U(t)\right\}KU(-t') \\
&\quad + U_0(t)K_0\left\{U_0(-T_1)U(T_1)-U_0(-t')U(t')\right\}[K-\mathbb{1}]U(-t') \\
&= -U_0(t)U_0(-t)U(t)KU(-t')-U_0(t)[K_0-\mathbb{1}]U_0(-T_0)U(T_0)KU(-t') \\
&\quad + U_0(t)K_0U_0(-T_1)U(T_1)[K-\mathbb{1}]U(-t')+U_0(t)K_0U_0(-t')U(t')U(-t') \\
&= -\frac{1}{\gamma}G(t,t')+\frac{1}{\gamma}G_0(t,t')-\frac{1}{\gamma^2}G_0(t,T_0)G(T_1,t')+\frac{1}{\gamma^2}G_0(t,T_1)G(T_1,t').
\end{aligned}$$

This yields (2.31) for $t \le t'$. Let $t > t'$. We again split up the integral:

$$\int_{T_0}^{T_1} = \int_{T_0}^{t'} + \int_{t'}^{t} + \int_{t}^{T_1}$$

and insert the definitions of G and G_0:

$$\begin{aligned}
\frac{1}{\gamma^2}\int_{T_0}^{T_1} G_0(t,\tau)S_1G(\tau,t')\,d\tau &= \int_{T_0}^{t'} U_0(t)[K_0-\mathbb{1}]U_0(-\tau)BU(\tau)KU(-t')\,d\tau \\
&\quad + \int_{t'}^{t} U_0(t)[K_0-\mathbb{1}]U_0(-\tau)BU(\tau)[K-\mathbb{1}]U(-t')\,d\tau \\
&\quad + \int_{t}^{T_1} U_0(t)K_0U_0(-\tau)BU(\tau)[K-\mathbb{1}]U(-t')\,d\tau.
\end{aligned}$$

The integration rule from Theorem 2.32 yields analogously:

$$\begin{aligned}
&\frac{1}{\gamma^2}\int_{T_0}^{T_1} G_0(t,\tau)S_1G(\tau,t')\,d\tau \\
&= U_0(t)[K_0-\mathbb{1}]\big(U_0(-t')U(t')-U_0(-T_0)U(T_0)\big)KU(-t') \\
&\quad + U_0(t)[K_0-\mathbb{1}]\big(U_0(-t)U(t)-U_0(-t')U(t')\big)[K-\mathbb{1}]U(-t') \\
&\quad + U_0(t)K_0\left(U_0(-T_1)U(T_1)-U_0(-t)U(t)\right)[K-\mathbb{1}]U(-t') \\
&= -U_0(t)[K_0-\mathbb{1}]U_0(-T_0)U(T_0)KU(-t')+U_0(t)[K_0-\mathbb{1}]U_0(-t')U(t')U(-t') \\
&\quad - U_0(t)U_0(-t)U(t)[K-\mathbb{1}]U(-t')+U_0(t)K_0U_0(-T_1)U(T_1)[K-\mathbb{1}]U(-t') \\
&= \frac{1}{\gamma}G(t,t')-\frac{1}{\gamma}G_0(t,t')-\frac{1}{\gamma^2}G_0(t,T_0)G(T_0,t')+\frac{1}{\gamma^2}G_0(t,T_1)G(T_1,t').
\end{aligned}$$

This yields (2.31) for für $t > t'$.

For $T_0 = -\infty$ because of the assumption (2.32) we can employ formula (2.20) from Theorem 2.32. □

3 Operators with dichotomy

3.1 Motivation

We consider linear operators, the spectrum of which has a special structure.

Definition 3.1. *Let $S : D(S) \to \mathcal{H}$ be a closed operator. We call S dichotomous or having a dichotomy (with respect to to the real axis[1]) if its spectrum contains no real values.*

We shall consider operators that are decomposed into real and imaginary part: $S = S_r + iS_i$ with symmetric operators S_r and S_i. In order to see what can happen we start with a simple example. Let $\mathcal{H} = \mathbb{C}^2$ and:

$$A(V) := \begin{pmatrix} \lambda_1 & V \\ \bar{V} & \lambda_2 \end{pmatrix}, \ \lambda_1, \lambda_2 \in \mathbb{R}, \ V \in \mathbb{C}.$$

We ask whether the spectrum of $A(V)$ can be shifted into the complex plane by perturbing the diagonal entries of $A(V)$. If we add a multiple of the unit matrix this question can be answered satisfactorily as we shall see in Theorem 3.2. Thus let

$$A_\varepsilon(V) := \begin{pmatrix} \lambda_1 + i\varepsilon & V \\ \bar{V} & \lambda_2 - i\varepsilon \end{pmatrix}, \ \varepsilon \in \mathbb{R}.$$

For $\varepsilon \neq 0$ obviously $A_\varepsilon(0)$ has no real eigenvalues. The charactaristic equation for $A_\varepsilon(V)$ reads:

$$\mu^2 - (\lambda_1 + \lambda_2)\mu + (\lambda_1 + i\varepsilon)(\lambda_2 - i\varepsilon) - |V|^2 = 0$$

having the solutions

$$\mu_{1,2} = \frac{\lambda_1 + \lambda_2}{2} \pm \sqrt{r + i\varepsilon(\lambda_1 - \lambda_2)}, \ r := \frac{(\lambda_1 - \lambda_2)^2}{4} - \varepsilon^2 + |V|^2.$$

It is easily checked that:

$$\operatorname{Im} \mu_{1,2} = \pm \frac{1}{\sqrt{2}} \left(-r + \sqrt{r^2 + \varepsilon^2(\lambda_1 - \lambda_2)^2} \right)^{1/2}.$$

$\lambda_1 \neq \lambda_2$ implies $\operatorname{Im} \mu_{1,2} \neq 0$ for all $\varepsilon \neq 0$. In case $\lambda_1 = \lambda_2$ this is no longer correct because for sufficiently small ε we have:

$$r = |V|^2 - \varepsilon^2 \geq 0$$

thus leading to:

$$\operatorname{Im} \mu_{1,2} = \pm \frac{1}{\sqrt{2}} (-r + \sqrt{r^2})^{1/2} = \pm \frac{1}{\sqrt{2}} (-r + r)^{1/2} = 0.$$

This shows that the 'gap', i.e., the difference between λ_1 and λ_2 counts rather than the smallness of the perturbation V in order to avoid real eigenvalues of $A_\varepsilon(V)$. We want to extend this result to general Hilbert spaces.

[1] In the literature dichotomy sometimes is defined with respect to to the imaginary axis

3.2 Dichotomy criteria

The most simple dichomoty criterion is achieved if we shift the spectrum of a self-adjoint operator by the same imaginary part.

Theorem 3.2. *Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator. Then the resolvent $((x+i\varepsilon)\mathbb{1} - A)^{-1}$ exists for all $\varepsilon \neq 0$ and all $x \in \mathbb{R}$ and is bounded with bound:*

$$\|((x+i\varepsilon)\mathbb{1} - A)^{-1}\| \leq \frac{1}{|\varepsilon|} \tag{3.1}$$

Proof. Let $\varphi \in D(A)$. We have:

$$\begin{aligned}\|((x+i\varepsilon)\mathbb{1} - A)\varphi\|^2 &= \|(x\mathbb{1} - A)\varphi\|^2 + \varepsilon^2\|\varphi\|^2 + 2\operatorname{Re} i\varepsilon((x\mathbb{1} - A)\varphi, \varphi) \\ &= \|(x\mathbb{1} - A)\varphi\|^2 + \varepsilon^2\|\varphi\|^2 \\ &\geq \varepsilon^2\|\varphi\|^2.\end{aligned}$$

This gives the statement with the aid of standard arguments (see [12]). □

Theorem 3.2 is the well known fact that a self-adjoint operator has a purely real spectrum. If we add an arbitrary imaginary part we can prove by similar calculations a stronger version of this result going back in somewhat more general form to Baskakov [6].

Theorem 3.3 (Baskakov). *Let $A, K : \mathcal{H} \to \mathcal{H}$ be bounded self-adjoint operators. Let K have a bounded inverse. If furthermore*

$$\|[A, K]\| < \frac{1}{\|K^{-1}\|^2}$$

then $A + iK$ is dichotomous.

Proof. For $\varphi \in \mathcal{H}$ and $x \in \mathbb{R}$ we have:

$$\begin{aligned}\|(x\mathbb{1} - A - iK)\varphi\|^2 &= \|(x\mathbb{1} - A)\varphi\|^2 + \|K\varphi\|^2 - 2\operatorname{Re} i((x\mathbb{1} - A)\varphi, K\varphi) \\ &= \|(x\mathbb{1} - A)\varphi\|^2 + \|K\varphi\|^2 + i(\varphi, [A, K]\varphi) \\ &\geq \frac{1}{\|K^{-1}\|^2}\|\varphi\|^2 - \|[A, K]\| \cdot \|\varphi\|^2 \\ &= \left(\frac{1}{\|K^{-1}\|^2} - \|[A, K]\|\right)\|\varphi\|^2.\end{aligned}$$

Together with the assumption this yields the theorem. □

In connection with the integral formula in section 5 operators of a special form are important. Let $A : D(A) \to \mathcal{H}$ be self-adjoint and $P : \mathcal{H} \to \mathcal{H}$ be an orthogonal projection having a finite dimensional range. Then we ask whether

$$A_\varepsilon := A + i\varepsilon(2P - \mathbb{1}), \ \varepsilon \in \mathbb{R} \setminus \{0\} \tag{3.2}$$

has a dichotomy.[2] Here the difficulty is that in the integral formula we have to perform the limit $\varepsilon \to 0$. Therefore we cannot apply the criterion of Baskakov assuming a large imaginary part. Theorem 3.2 suggests to consider the operator on the subspaces $\operatorname{ran} P$ and $(\operatorname{ran} P)^\perp$. We prove a simple algebraic invertibility criterion.

[2] Roughly speaking one tries to shift one part of the spectrum into the upper half plane and the other one into the lower half plane.

Lemma 3.4. *Let D_1, D_2 and $\mathcal{H}_1$, $\mathcal{H}_2$ be linear spaces and $S : D_1 \oplus D_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$ a linear operator. We write S as:*

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \quad S_{jk} : D_k \to \mathcal{H}_j, \; j,k = 1,2.$$

Let the operator $S_{22} : D_2 \to \mathcal{H}_2$ be invertible. Then S has an inverse if and only if

$$(S_{11} - S_{12}S_{22}^{-1}S_{21})^{-1} : \mathcal{H}_1 \to D_1$$

exists.

Proof. Let S be invertible. We write the inverse operator $\tilde{S} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to D_1 \oplus D_2$ as

$$\tilde{S} := \begin{pmatrix} \tilde{S}_{11} & \tilde{S}_{12} \\ \tilde{S}_{21} & \tilde{S}_{22} \end{pmatrix}, \quad \tilde{S}_{jk} : \mathcal{H}_k \to D_j, \; j,k = 1,2,$$

and compute the product $S\tilde{S}$:

$$S\tilde{S} = \begin{pmatrix} S_{11}\tilde{S}_{11} + S_{12}\tilde{S}_{21} & S_{11}\tilde{S}_{12} + S_{12}\tilde{S}_{22} \\ S_{21}\tilde{S}_{11} + S_{22}\tilde{S}_{21} & S_{21}\tilde{S}_{12} + S_{22}\tilde{S}_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}.$$

Comparing the entries of the first column gives:

$$\tilde{S}_{21} = -S_{22}^{-1}S_{21}\tilde{S}_{11} \tag{3.3}$$

$$(S_{11} - S_{12}S_{22}^{-1}S_{21})\tilde{S}_{11} = \mathbb{1}. \tag{3.4}$$

This shows the existence of $(S_{11} - S_{12}S_{22}^{-1}S_{21})^{-1}$.

On the other hand if one defines $\tilde{S}_{11}$ and $\tilde{S}_{21}$ via (3.3) and (3.4) and $\tilde{S}_{12}$ and $\tilde{S}_{22}$ analogously we see that S is invertible. □

From the preceding lemma we can derive a dichotomy criterion sufficient for our purposes.

Theorem 3.5. *Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator and $P : \mathcal{H} \to \mathcal{H}$ be an orthogonal projection having a finite dimensional range and the property $P\varphi \in D(A)$ for all $\varphi \in \mathcal{H}$. We write $D_1 := \mathcal{H}_1 := \operatorname{ran} P$, $\mathcal{H}_2 := \mathcal{H}_1^{\perp}$, and $D_2 := D(A) \cap \mathcal{H}_2$. Let D_2 be dense in $\mathcal{H}_2$ and the operator $(\mathbb{1} - P)A(\mathbb{1} - P) : D_2 \to \mathcal{H}_2$ be self-adjoint. Define*

$$\lambda := \sup_{\substack{\varphi \in \mathcal{H}_1 \\ \|\varphi\|=1}} (\varphi, A\varphi) \text{ and } \mu := \inf_{\substack{\varphi \in D_2 \\ \|\varphi\|=1}} (\varphi, A\varphi).$$

If $\lambda < \mu$ then $A_\varepsilon := A + i\varepsilon(2P - \mathbb{1})$ is dichotomous for all $\varepsilon \in \mathbb{R} \setminus \{0\}$.

Proof. We write A_ε as block matrix:

$$A_\varepsilon = \begin{pmatrix} A_{11} + i\varepsilon\mathbb{1} & A_{12} \\ A_{21} & A_{22} - i\varepsilon\mathbb{1} \end{pmatrix}, \quad A_{jk} : D_k \to \mathcal{H}_j, \; j,k = 1,2.$$

We do not distinguish between the different identity operators acting on different subspaces. By assumption $A_{22} : D_2 \to \mathcal{H}_2$ is self-adjoint. Therefore by Theorem 3.2 the resolvent

$(A_{22} - i\varepsilon\mathbb{1} - x\mathbb{1})^{-1}$ exists and the conditions of Lemma 3.4 are satisfied. Thus $(A_\varepsilon - x\mathbb{1})^{-1} : \mathcal{H} \to D$ exists if and only if

$$Z := A_{11} - x\mathbb{1} + i\varepsilon\mathbb{1} - A_{12}(A_{22} - x\mathbb{1} - i\varepsilon\mathbb{1})^{-1}A_{21},\ Z : \mathcal{H}_1 \to \mathcal{H}_1,$$

is invertible. Since $\mathcal{H}_1$ is finite dimensional it suffices to show injectivity. To this end let $\varphi \in \mathcal{H}_1$ with $Z\varphi = 0$. We split Z into a sum of symmetric and antisymmetric operators where we notice $A_{12}^* = A_{21}$:

$$\begin{aligned}
&A_{12}(A_{22} - (x + i\varepsilon)\mathbb{1})^{-1}A_{21}\\
&= A_{12}(A_{22} - (x - i\varepsilon)\mathbb{1})^{-1}(A_{22} - (x - i\varepsilon)\mathbb{1})(A_{22} - (x + i\varepsilon)\mathbb{1})^{-1}A_{21}\\
&= A_{12}(A_{22} - (x - i\varepsilon)\mathbb{1})^{-1}(A_{22} - x\mathbb{1})(A_{22} - (x + i\varepsilon)\mathbb{1})^{-1}A_{21}\\
&\quad + i\varepsilon A_{12}(A_{22} - (x - i\varepsilon)\mathbb{1})^{-1}(A_{22} - (x + i\varepsilon)\mathbb{1})^{-1}A_{21}
\end{aligned}$$

and consider the quadratic form with φ:

$$\begin{aligned}
0 &= (\varphi, Z\varphi)\\
&= (\varphi, (A_{11} - x\mathbb{1})\varphi) + i\varepsilon\|\varphi\|^2\\
&\quad - (\varphi, A_{12}(A_{22} - (x - i\varepsilon)\mathbb{1})^{-1}(A_{22} - x\mathbb{1})(A_{22} - (x + i\varepsilon)\mathbb{1})^{-1}A_{21}\varphi)\\
&\quad - i\varepsilon(\varphi, A_{12}(A_{22} - (x - i\varepsilon)\mathbb{1})^{-1}(A_{22} - (x + i\varepsilon)\mathbb{1})^{-1}A_{21}\varphi)\\
&= (\varphi, (A_{11} - x\mathbb{1})\varphi) - ((A_{22} - (x + i\varepsilon)\mathbb{1})^{-1}A_{21}\varphi, (A_{22} - x\mathbb{1})(A_{22} - (x + i\varepsilon)\mathbb{1})^{-1}A_{21}\varphi)\\
&\quad + i\varepsilon\|\varphi\|^2 - i\varepsilon\|(A_{22} - (x + i\varepsilon)\mathbb{1})^{-1}A_{21}\varphi\|^2.
\end{aligned}$$

Taking the real part gives if we notice the definition of λ and μ:

$$\begin{aligned}
0 &= (\varphi, (A_{11} - x\mathbb{1})\varphi) - ((A_{22} - (x + i\varepsilon)\mathbb{1})^{-1}A_{21}\varphi, (A_{22} - x\mathbb{1})(A_{22} - (x + i\varepsilon)\mathbb{1})^{-1}A_{21}\varphi)\\
&\le (\lambda - x)\|\varphi\|^2 - (\mu - x)\|(A_{22} - (x + i\varepsilon)\mathbb{1})^{-1}A_{21}\varphi\|^2.
\end{aligned}$$

The imaginary part can be used to rewrite the second summand:

$$0 \le (\lambda - x)\|\varphi\|^2 - (\mu - x)\|\varphi\|^2 = (\lambda - \mu)\|\varphi\|^2.$$

Since $\lambda - \mu < 0$ this is only possible with $\|\varphi\| = 0$, i.e., $\varphi = 0$. $\square$

It is remarkable that Theorem 3.5 neither depends on the size of the off-diagonal elements nor on the size of the 'gap' between A_{11} and A_{22} as long as it differs from zero.

4 Section determinants of groups of bounded operators

4.1 The integral formula

We motivate our following examinations by considering a simple example for which all questions can be answered explicitly. We start with the following Schrödinger equation:

$$i\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} \lambda_1 & V \\ \bar{V} & \lambda_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\ u_1(0) = 1,\ u_2(0) = 0$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$ and $V \in \mathbb{C}$. We are interested in u_1. Therefore we decouple the system and obtain:

$$u_1'' + i(\lambda_1 + \lambda_2)u_1' - (\lambda_1\lambda_2 - |V|^2)u_1 = 0,\ u_1(0) = 1,\ u_1'(0) = -i\lambda_1.$$

The usual ansatz together with the initial conditions leads to the solution:

$$u_1(t) = \frac{e^{-i\frac{\lambda_1+\lambda_2}{2}t}}{2\omega}\left((\omega - \frac{\Lambda}{2})e^{i\omega t} + (\omega + \frac{\Lambda}{2})e^{-i\omega t}\right)$$

with

$$\Lambda := \lambda_1 - \lambda_2,\ \omega := \sqrt{\frac{\Lambda^2}{4} + |V|^2}.$$

This expression already shows the characteristic feature of the integral formula since essentially the solution of the unperturbed problem is multiplied by a correcting factor. We rewrite the solution into another form:

$$u_1(t) = \frac{e^{-i\frac{\lambda_1+\lambda_2}{2}t}}{2\omega}\left(2\omega\cos\omega t - i\Lambda\sin\omega t\right).$$

This shows $u_1(t) \neq 0$ for all t if only $\Lambda \neq 0$ (for $\omega = 0$ everything becomes trivial). We even have a quantitative statement:

$$\begin{aligned} |u_1(t)|^2 &= \frac{1}{4\omega^2}\left(4\omega^2\cos^2\omega t + \Lambda^2\sin^2\omega t\right) \\ &= \cos^2\omega t + \frac{\Lambda^2}{\Lambda^2 + 4|V|^2}\sin^2\omega t \\ &= 1 - \frac{4|V|^2}{\Lambda^2 + 4|V|^2}\sin^2\omega t. \end{aligned}$$

Thus, if $|\Lambda|$ is large compared with $|V|$ we have:

$$|u_1(t)|^2 = 1 + O(\frac{|V|^2}{\Lambda^2}).$$

We want to examine similar questions in arbitrary Hilbert spaces. We do not restrict ourselves to unitary operators but prove more generally an integral formula for the section determinants of groups of bounded operators.

Theorem 4.1. *Let the operator $A_0 : D(A_0) \to \mathcal{H}$ be densely defined and closed and let $-iA_0$ generate a group $U_0(t)$. Let $B : \mathcal{H} \to \mathcal{H}$ be a bounded operator. We define the operators $A(\alpha) : D(A_0) \to \mathcal{H}$ by $A(\alpha) := A_0 + \alpha B$, $\alpha \in [0,1]$. Moreover let $P : \mathcal{H} \to \mathcal{H}$ be a orthogonal projection with finite dimensional range commuting with $U_0(t)$: $PU_0(t) = U_0(t)P$. Let $U(t,\alpha)$ be the group generated by $-iA(\alpha)$. We require* $\det PU(T,\alpha)P \neq 0$, $\alpha \in [0,1]$. *Finally let the free (time-ordered) Green operator G_0 be defined by:*

$$G_0(t) = \begin{cases} iU_0(t)P = iPU_0(t), & t \leq 0, \\ iU_0(t)[P - \mathbb{1}] = i[P - \mathbb{1}]U_0(t), & t > 0. \end{cases} \tag{4.1}$$

Then the integral equation

$$G(t,t';\alpha)\varphi = G_0(t-t')\varphi + \alpha \int_0^T G_0(t-\tau)BG(\tau,t';\alpha)\varphi\, d\tau,$$
$$t,t' \in [0,T],\ \alpha \in [0,1],\ \varphi \in \mathcal{H}, \tag{4.2}$$

possesses (at least) one solution G, the time-ordered Green operator, having the properties:

1. *The function $t \mapsto G(t,t';\alpha)$ is strongly continuous for $t \neq t'$ and all $t' \in \mathbb{R}$ and all $\alpha \in [0,1]$.*

2. *The operator $G(t,t+0;\alpha)$ is trace class and the function* $\operatorname{tr} BG(t,t+0;\alpha)$ *is integrable with respect to t and α such that for the section determinants there holds the integral formula:*

$$\det PU(T,1)P = \det PU_0(T)P \exp\left[-\int_0^1 \int_0^T \operatorname{tr} BG(t,t+0;\alpha)\, dt\, d\alpha\right]. \tag{4.3}$$

This theorem has a perturbation theoretic character. It relates the determinant belonging to the perturbed operator $A_0 + B$ to the determinant belonging to the unperturbed operator A_0. The main idea in proving Theorem 4.1 is to study the dependence on the parameter α by calculating the derivative. Contrary to this one expands in the classical perturbation theory as we shall demonstrate in Section 6.2 all interesting quantities in a power series in α. Such an expansion can also be derived from Theorem 4.1 by, for example, iterating the integral equation (4.2). But this is just one possibility among others and not that one that should be recommended.

We work with the group generated by $-iA(\alpha)$ rather than by $A(\alpha)$ in order to keep the formal relationship with the Schrödinger equation and the following section concerning Riesz projections.

We divide the proof of 4.1 into two steps.

Step 1: We apply the determinant formula in Corollary 2.6 to the finite dimensional operator $PU(T,\alpha)P : \operatorname{ran} P \to \operatorname{ran} P$:

$$\det PU(T,1)P = \det PU_0(T)P \exp\left[\int_0^1 \operatorname{tr} U^+(T,\alpha)\frac{\partial U(T,\alpha)}{\partial \alpha}\, d\alpha\right].$$

Here $U^+(T,\alpha)$ is the pseudo inverse of $U(T,\alpha)$ with respect to P according to Theorem 2.8. The derivative $\frac{\partial U(T,\alpha)}{\partial\alpha}$ was computed in Theorem 2.35. We thus arrive at the formula:

$$\det PU(T)P = \det PU_0(T)P \exp\left[-i\int_0^1\int_0^T \operatorname{tr} BU(t,\alpha)U^+(T,\alpha)U(T,\alpha)U(-t,\alpha)\,dt\right]. \tag{4.4}$$

Step 2: We define the operator G in the integral formular (4.3) essentially as the exponent in (4.4):

$$G(t,t;\alpha) := iU(t,\alpha)U^+(T,\alpha)U(T,\alpha)U(-t,\alpha)$$

and derive the integral equation (4.2) for $G(t,t';\alpha)$ where we use the integral equation (2.33) for general Green operators from Theorem 2.42.

4.2 Proof of the integral formula

We assumed that the operator $B:\mathcal{H}\to\mathcal{H}$ is bounded. Therefore it follows at once that $A(\alpha)$ is defined on $D(A_0)$. Furthermore we know from Lemma 2.31 that $-iA(\alpha)$ generates a group $U(t,\alpha)$ because $-iA_0$ is the generator of a group.

4.2.1 Step 1: The finite dimensional determinant formula

We apply (2.4) to the finite dimensional operator $PU(t,\alpha)P:\operatorname{ran}P\to\operatorname{ran}P$.

Theorem 4.2. *Let $U^+(T,\alpha)$ be the pseudo inverse of $U(T,\alpha)$ with respect to P. Then we have:*

$$\det PU(T,1)P = \det PU_0(T)P \exp\left[-i\int_0^1\int_0^T \operatorname{tr} BU(t,\alpha)U^+(T,\alpha)U(T,\alpha)U(-t,\alpha)\,dt\,d\alpha\right].$$

Proof. By Theorem 2.35 $U(T,\alpha)$ is continuously differentiable with respect to α in the operator norm. Together with $\det PU(T,\alpha)P\neq 0$ the finite dimensional operator $PU(T,\alpha)P:\operatorname{ran}P\to\operatorname{ran}P$ satisfies the assumptions of Corollary 2.6. Since $U(T,0)=U_0(T)$ it follows:

$$\det PU(T,1)P = \det PU_0(T)P\exp\left[\int_0^1 \operatorname{tr} U^+(T,\alpha)\frac{\partial U(T,\alpha)}{\partial\alpha}\,d\alpha\right].$$

The derivative $\partial U/\partial\alpha$ was computed in Theorem 2.35:

$$\frac{\partial}{\partial\alpha}U(T,\alpha)\varphi = -i\int_0^T U(T,\alpha)U(-t,\alpha)BU(t,\alpha)\varphi\,dt,\ \varphi\in\mathcal{H}.$$

Since $U(t,\alpha)$ depends strongly continuously on t and the trace refers only to the finite dimensional space $\operatorname{ran}P$ the following integrals are well defined (Theorem 2.3). Noting the cyclic commutativity under the trace it follows:

$$\begin{aligned}\operatorname{tr} U^+(T,\alpha)\frac{\partial U(T,\alpha)}{\partial\alpha} &= \int_0^T \operatorname{tr} U^+(T,\alpha)[-iU(T,\alpha)U(-t,\alpha)BU(t,\alpha)]\,dt\\ &= \int_0^T \operatorname{tr} BU(t,\alpha)U^+(T,\alpha)[-iU(T,\alpha)U(-t,\alpha)]\,dt.\end{aligned}$$

This yields the statement. □

4.2.2 Step 2: The integral equation

The expression $iU(t,\alpha)U^+(T,\alpha)U(T,\alpha)U(-t,\alpha)$ can be described independently of $U(t,\alpha)$. To do that we need the Green operators from Section 2.4. We set $\gamma = i$ and $K = U^+(T,\alpha)U(T,\alpha)$ in Definition 2.40.

Definition 4.3. *Let $U^+(T,\alpha)$ be the pseudo inverse of $U(T,\alpha)$ with respect to P. The operator defined by*

$$G(t,t';\alpha) := \begin{cases} iU(t,\alpha)U^+(T,\alpha)U(T,\alpha)U(-t',\alpha), & t \le t', \\ iU(t,\alpha)[U^+(T,\alpha)U(T,\alpha) - \mathbb{1}]U(-t',\alpha), & t > t', \end{cases}$$

is called the time-ordered Green operator.[1]

We note the properties of G.

Lemma 4.4. *Let G be the time-ordered Green operator according to Definition 4.3. Then the map $t \mapsto G(t,t';\alpha)$ is strongly continuous for $t \neq t'$ and all $t' \in \mathbb{R}$. The operator $G(t,t+0;\alpha)$ is trace class for all $t \in \mathbb{R}$ and all $\alpha \in [0,1]$ and the following integral exists:*

$$\int_0^1 \int_0^T \operatorname{tr} BG(t,t+0;\alpha)\,dt\,d\alpha.$$

Proof. The strong continuity of $G(\cdot,t';\alpha)$ (for $t \neq t'$) follows immediately from the strong continuity of $U(t,\alpha)$ and the definition of G.

Since $U^+(T,\alpha) = PU^+(T,\alpha)$ has finite dimensional range and thus is trace class, $G(t,t+0;\alpha)$ is trace class, too.

The existence of the integral follows immediately from Theorem 4.2. □

G satisfies simple boundary conditions.

Lemma 4.5. *The time-ordered Green operator according to Definition 4.3 satisfies for all $\alpha \in [0,1]$ the boundary conditions:*

$$(\mathbb{1} - P)G(0,t';\alpha) = 0, \;\; PG(T,t';\alpha) = 0, \;\; 0 \le t' \le T. \tag{4.5}$$

Proof. We note $U^+(T,\alpha) = PU^+(T,\alpha)$ (Theorem 2.8):

$$\begin{aligned}(\mathbb{1} - P)G(0,t';\alpha) &= i(\mathbb{1} - P)U(0,\alpha)U^+(T,\alpha)U(T,\alpha)U(-t',\alpha) \\ &= i(\mathbb{1} - P)U^+(T,\alpha)U(T,\alpha)U(-t',\alpha) \\ &= 0.\end{aligned}$$

For $t = T$ we use $PU(T,\alpha)U^+(T,\alpha) = P$ (see Theorem 2.8):

$$\begin{aligned}PG(T,t';\alpha) &= iPU(T,\alpha)[U^+(T,\alpha)U(T,\alpha) - \mathbb{1}]U(-t',\alpha) \\ &= i[PU(T,\alpha) - PU(T,\alpha)]U(-t',\alpha) \\ &= 0.\end{aligned}$$

This proves the theorem. □

[1] Of course, G depends on the intervall $[0,T]$. We do not indicate this because otherwise we had too much variables. In the physics literature the solution to Schrödinger equations with time dependent potentials is written as a formal exponential series, the so-called time-ordered exponential function. This may explain the name time-ordered Green operator.

For $\alpha = 0$ we recover the free time-ordered Green operator.

Lemma 4.6. *For the free time-ordered Green operator $G_0(t)$ from (4.1) we have: $G(t,t';0) = G_0(t-t')$. If A_0 is self-adjoint we have $\|G_0(t)\| \le 1$ for all $t \in \mathbb{R}$.*

Proof. By assumption P commutes with $U_0(t)$. Thus:

$$U_0^+(T) = U_0(-T)P.$$

Hence we have for $t \le t'$:

$$-iG(t,t';0) = U_0(t)U_0^+(T)U_0(T)U_0(-t') = U_0(t)PU_0(-t') = U_0(t-t') = -iG_0(t-t').$$

For $t > t'$ the calculations are analogous whence the first part of the statement follows.

If A_0 is self-adjoint then $U_0(t)$ is unitary by Corollary 2.24. Since P is an orthogonal projection we have $\|P\| \le 1$. For $t \le 0$ it follows:

$$\|G_0(t)\| = \|iPU_0(t)\| \le \|P\|\|U_0(t)\| \le 1.$$

The statement for $t > 0$ can be obtained in a similar way because of $\|\mathbb{1} - P\| \le 1$. □

The time-ordered Green operator G and the free time-ordered Green operator G_0 are related through an integral equation.

Theorem 4.7. *The time-ordered Green operator G according to Definition 4.3 satisfies the integral equation:*

$$G(t,t';\alpha)\varphi = G_0(t-t')\varphi + \alpha \int_0^T G_0(t-\tau)BG(\tau,t';\alpha)\varphi\,d\tau,$$
$$0 \le t,t' \le T,\ \alpha \in [0,1],\ \varphi \in \mathcal{H}. \quad (4.6)$$

Proof. We set $\gamma = i$, $S_1 = -iB$ and $T_0 = 0$, $T_1 = T$ in Theorem 2.42 and get:

$$\begin{aligned} G(t,t';\alpha)\varphi + G_0(t)G(0,t';\alpha)\varphi - G_0(t-T)G(T,t';\alpha)\varphi \\ = G_0(t-t')\varphi + \alpha \int_0^T G_0(t-\tau)BG(\tau,t';\alpha)\varphi\,d\tau,\ \varphi \in \mathcal{H}. \end{aligned}$$

This is almost the desired integral equation. We recall the definition of G_0 and note $0 \le t \le T$ and the boundary conditions (4.5) for G:

$$\begin{aligned} G_0(t)G(0,t';\alpha) &= iU_0(t)[P - \mathbb{1}]G(0,t';\alpha) = 0 \\ G_0(t-T)G(T,t';\alpha) &= iU_0(t-T)PG(T,t';\alpha) = 0. \end{aligned}$$

This yields the theorem. □

Now Theorem 4.1 is proved by combining Theorems 4.2 and 4.7 with Definition 4.3.

4.3 Solution theory of the integral equation

We use the notations and assumptions of Theorem 4.1.[2] We examine the relation between the unique solvability of the integral equation (4.2) and the determinant det $PU(T,\alpha)P$. We shall show that uniqueness of the solution already implies existence of a solution for the special right hand side in (4.2). Hence we only need to consider the homogeneous integral equation. First of all we show that the homogeneous euqation is connected with a homogeneous boundary value problem.

Theorem 4.8. *Let the function $\varphi(\cdot): t \mapsto \varphi(t) \in \mathcal{H}$ be continuously differentiable and solve the integral equation:*

$$\varphi(t) = \alpha \int_0^T G_0(t-\tau)B\varphi(\tau)\,d\tau,\ 0 \le t \le T. \tag{4.7}$$

Then $\varphi(\cdot)$ is also a solution to the boundary value problem:

$$i\frac{d}{dt}\varphi(t) = A(\alpha)\varphi(t),\ (\mathbb{1}-P)\varphi(0) = 0,\ P\varphi(T) = 0. \tag{4.8}$$

Proof. We split up the integral in (4.7):

$$I(t) := \int_0^T G_0(t-\tau)B\varphi(\tau)\,d\tau = \int_0^t iU_0(t-\tau)[P-\mathbb{1}]B\varphi(\tau)\,d\tau + \int_t^T iU_0(t-\tau)PB\varphi(\tau)\,d\tau.$$

We have assumed that $\varphi(\cdot)$ is continuously differentiable. Then the functions $\tau \mapsto [P-\mathbb{1}]B\varphi(\tau)$ and $\tau \mapsto PB\varphi(\tau)$ are also continuously differentiable since P and B are bounded. From Theorem 2.26 it follows that $I(\cdot)$ is continuously differentiable and that $\varphi(\cdot)$ solves the differential equation (4.8). For the free time-ordered Green operator G_0 we have by definition for all $\tau \in [0,T]$:

$$(\mathbb{1}-P)G_0(-\tau) = i(\mathbb{1}-P)PU_0(-\tau) = 0,\ PG_0(T-\tau) = iP(P-\mathbb{1})U_0(T-\tau) = 0.$$

This proves the boundary conditions. ☐

The assumption that the solution to the integral equation is continuously differentiable generally does not follow from the integral equation when the generator is unbounded (see [11], Section 4.2.) and therefore cannot be dropped. But for bounded operators this is correct.

Corollary 4.9. *In addition to the assumptions of Theorem 4.1 let $A_0 : \mathcal{H} \to \mathcal{H}$ be bounded. Then every continuous solution to the integral equation (4.7) is differentiable and satisfies the boundary value problem (4.8).*

Proof. The group $U_0(t)$ is generated by the bounded operator $-iA_0$ and thus is differentiable in the operator norm (Theorem 2.27). Then the statement follows from the proof of Theorem 4.8. ☐

Theorem 4.8 has a converse.

[2] In this section the parameter α could be allowed to take arbitrary complex values. But this is not necessary for our purposes.

Theorem 4.10. *Let the function $\varphi(\cdot) : t \mapsto \varphi(t) \in \mathcal{H}$ be continuously differentiable and let $\varphi(0) \in D(A)$. Then if $\varphi(\cdot)$ solves the boundary value problem (4.8) $\varphi(\cdot)$ is also a solution to the integral equation (4.7).*

Proof. Let $\varphi(\cdot)$ be a solution to (4.8). By Theorem 2.25 $\varphi(t)$ can be written as

$$\varphi(t) = U(t, \alpha)\varphi(0)$$

since $\varphi(0) \in D(A)$. From the integration rule (2.17) we get for $t_0 = 0$ and $t_1 = t$:

$$U(t, \alpha)\varphi(0) = U_0(t)\varphi(0) - i\alpha \int_0^t U_0(t - \tau)BU(\tau, \alpha)\varphi(0)\, d\tau. \tag{4.9}$$

For $t_0 = t$ and $t_1 = T$, (2.17) yields:

$$U(t, \alpha)\varphi(0) = U_0(t - T)U(T, \alpha)\varphi(0) + i\alpha \int_t^T U_0(t - \tau)BU(\tau, \alpha)\varphi(0)\, d\tau. \tag{4.10}$$

We multiply (4.9) by $\mathbb{1} - P$ and (4.10) by P and notice $PU_0(t) = U_0(t)P$. Then together with the boundary conditions it follows:

$$(\mathbb{1} - P)\varphi(t) = 0 - i\alpha \int_0^t (\mathbb{1} - P)U_0(t - \tau)B\varphi(\tau)\, d\tau \tag{4.11}$$

$$P\varphi(t) = 0 + i\alpha \int_t^T PU_0(t - \tau)B\varphi(\tau)\, d\tau. \tag{4.12}$$

Adding (4.11) and (4.12) gives the statement. □

The solvability of the boundary value problem can be characterized by $\det PU(T, \alpha)P$.

Theorem 4.11. *The boundary value problem (4.8) only has the trivial solution if and only if* $\det PU(T, \alpha)P \neq 0$.

Proof. By Theorem 2.25 the solution to the boundary value problem can be written as:

$$\varphi(t) = U(t, \alpha)\varphi(0).$$

From the boundary conditions it follows $\varphi(0) = P\varphi(0)$ and:

$$0 = P\varphi(T) = PU(T, \alpha)\varphi(0) = PU(T, \alpha)P\varphi(0).$$

Considering this equation on $\operatorname{ran} P$ shows that $\det PU(T, \alpha)P \neq 0$ is equivalent to $\varphi(0) = 0$. This proves the theorem. □

From the preceding considerations we get a criterion for the non-vanishing of the determinant $\det PU(T, \alpha)P$.

Corollary 4.12. *Let the homogeneous equation (4.7) only have the trivial solution. Then:*

1. $\det PU(T, \alpha)P \neq 0$.
2. *The inhomogeneous equation (4.2) has exactly one solution.*

Proof. 1. If $\det PU(T,\alpha)P = 0$ the boundary value problem (4.8) has a non-trivial solution, which is also a non trivial solution to the integral equation (4.7). This condradicts the assumption.

2. Since the homogeneous equation (4.7) only has the trivial solution the inhomogeneous equation (4.2) can have at most one solution. Since $\det PU(T,\alpha)P \neq 0$ by 1. in Definition 4.3 the operator $U^+(T,\alpha)$ exists. Thus by Theorem 4.7 $G(t,t';\alpha)$ provides a solution to the equation (4.2). □

The above solvability criterion reminds on the Riesz theory for compact operators in that uniqueness of the solution implies existence of a solution. But we have to note that the existence criterion only refers to the special right hand side in (4.2).

4.4 Remarks

A careful analysis of the proof in Section 4.2 shows that the integral formula (4.3) can be generalized in different ways.

The requirement that $U(t,\alpha)$ form a group is superfluous as can be seen from the definition of G. It is enough to have a semi-group and especially the existence of $U(T-t,\alpha)$ for $T \geq t$. Thus, our results carry over to the heat equation.

Another assumption that can be dropped without any computational difficulties is the time independence of A. Actually, the formula is correct for time dependent operators $A(t) = A_0 + B(t)$:

$$\det PU(T)P = \det PU_0(T)P \times \exp\left[-\int_0^1 \int_0^T \operatorname{tr} B(t)G(t,t+0;\alpha)\, dt\, d\alpha\right].$$

The exponent satisfies the corresponding integral equation:

$$G(t,t';\alpha)\varphi = G_0(t-t')\varphi + \alpha \int_0^T G_0(t-\tau)B(\tau)G(\tau,t';\alpha)\varphi\, d\tau,\ \ \varphi \in \mathcal{H}.$$

We can even start with general families of operators $A(t,\alpha)$ without changing the formal calculations. But this will increase the technicalities because in this case $U(t)$ is not a group. Thus we first have to check the invertability of $U(t)$. If we assume $A(t) : \mathcal{H} \to \mathcal{H}$ to be bounded for all $t \in \mathbb{R}$ there are almost no technical difficulties and all calculations carry over to this case. But applications in quantum mechanics require unbounded A_0. Then even for bounded $B(t)$ additional technical conditions are necessary.[3] The situation may be simplified somewhat if the time dependence only enters through a scalar factor:

$$B(t) = f(t)B,\ \ f : \mathbb{R} \to \mathbb{R}.$$

This is the form of operators appearing in the introduction in connection with the adiabatic theorem. It is possible to extend Theorem 4.1 to this case.

Furthermore one can allow unbounded time intervalls. This is important in order to provide a relation with the Riesz projections via the adiabatic theorem (see the introduction).

[3] General statements concerning such evolution equations may be found in [11] or [20].

5 Section determinants of Riesz projections

5.1 The integral formula

Let us begin with a simple example. Given a matrix

$$A(V) := \begin{pmatrix} \lambda_1 & V \\ \bar{V} & \lambda_2 \end{pmatrix}, \ \lambda_1, \lambda_2 \in \mathbb{R}, \ V \in \mathbb{C}$$

we are looking for a relation of the eigenvectors of $A(V)$ to those of $A(0)$. Elementary calculations show that

$$x_1(V) = \frac{\frac{\Lambda}{2} + \omega}{((\frac{\Lambda}{2} + \omega)^2 + |V|^2)^{1/2}}, \ x_2(V) = \frac{\bar{V}}{((\frac{\Lambda}{2} + \omega)^2 + |V|^2)^{1/2}}$$

is a normed eigenvector $(x_1(V), x_2(V))^T$ of $A(V)$. Here we have set

$$\Lambda := \lambda_1 - \lambda_2, \ \omega := \sqrt{\frac{\Lambda^2}{4} + |V|^2}.$$

Let $\Lambda > 0$. Then we get $x_1(0) = 1$ and $x_2(0) = 0$ so that $(x_1(V), x_2(V))^T$ corresponds to the unperturbed eigenvector $(1,0)^T$. The scalarproduct

$$\begin{pmatrix} x_1(V) \\ x_2(V) \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = x_1(V)$$

measures the angle between the two vectors. We observe $x_1(V) \neq 0$. But we also have an quantitative result:

$$|x_1(V)|^2 = \frac{(\frac{\Lambda}{2} + \omega)^2}{(\frac{\Lambda}{2} + \omega)^2 + |V|^2} = \frac{1}{1 + \frac{4|V|^2}{\Lambda^2\left[1 + \sqrt{1 + \frac{4|V|^2}{\Lambda^2}}\right]^2}}.$$

For small $\frac{|V|^2}{\Lambda^2}$ we get an expansion in powers of $\frac{|V|^2}{\Lambda^2}$. For $\Lambda = 0$ the situation differs from those in 3.1 (dichotomy) and 4.1 (section determinants of groups of bounded operators). Let $\Lambda = 0$, i.e $\lambda_1 = \lambda_2$. We still have $x_1(V) \neq 0$. But we cannot relate this eigenvector to the unperturbed one because it does not tend to $(1,0)^T$ as $V \to 0$. The reason is that for $V = 0$ we have a multiple eigenvalue.

In order to examine similar questions in general Hilbert spaces we prove an integral formula for Riesz projections of a self-adjoint operator.

Theorem 5.1. *Let $A_0 : D(A_0) \to \mathcal{H}$ be a self-adjoint operator and $B : \mathcal{H} \to \mathcal{H}$ a bounded symmetric operator. We define the operators $A(\alpha) : D(A_0) \to \mathcal{H}$, $\alpha \in [0,1]$, by $A(\alpha) := A_0 + \alpha B$. For all $\alpha \in [0,1]$ we require:*

1. *There is an integration path Γ with the property $\Gamma \subset \rho(A(\alpha))$. By $P(\alpha)$ we denote the Riesz projections of $A(\alpha)$ with respect to Γ:*

$$P(\alpha) := \frac{1}{2\pi i} \int_\Gamma (z\mathbb{1} - A(\alpha))^{-1}\, dz.$$

2. *The range of $P(0)$ is finite dimensional with $N := \dim \operatorname{ran} P(0) \geq 1$.*

3. $\det P(0)P(\alpha)P(0) \neq 0$.

4. *Let $\mathcal{H}_1 := \operatorname{ran} P(0) \subset D(A(\alpha))$ and $\mathcal{H}_2 := \mathcal{H}_1^{\perp}$. $D_2 := \mathcal{H}_2 \cap D(A_0)$ is a dense subspace of $\mathcal{H}_2$ and $(\mathbb{1} - P(0))A(\alpha)(\mathbb{1} - P(0)) : D_2 \to \mathcal{H}_2$ is self-adjoint. For*

$$\lambda(\alpha) := \sup_{\substack{\varphi \in \mathcal{H}_1 \\ \|\varphi\|=1}} (\varphi, P(0)A(\alpha)P(0)\varphi),\ \mu(\alpha) := \inf_{\substack{\varphi \in D_2 \\ \|\varphi\|=1}} (\varphi, (\mathbb{1} - P(0))A(\alpha)(\mathbb{1} - P(0))\varphi)$$

we require $\lambda(\alpha) < \mu(\alpha)$.

Finally let G_0 be the free time-ordered Green operator as in Theorem 4.1. Then for $\varepsilon > 0$ small enough the Wiener-Hopf equation

$$G_\varepsilon(t, t'; \alpha)\varphi = e^{-\varepsilon|t-t'|}G_0(t - t')\varphi + \alpha \int_{-\infty}^{0} e^{-\varepsilon|t-\tau|}G_0(t - \tau)BG_\varepsilon(\tau, t'; \alpha)\varphi \, d\tau,$$
$$-\infty < t, t' \leq 0,\ \alpha \in [0, 1],\ \varphi \in \mathcal{H}, \quad (5.1)$$

has (at least) one solution G_ε, the time-ordered Green operator, having the properties:

a. *The map $t \mapsto G_\varepsilon(t, t'; \alpha)$ is strongly continuous for $t \neq t'$ and all $t' \in \mathbb{R}$. We have the estimate $\|G_\varepsilon(t, t'; \alpha)\| \leq Ce^{\delta t}$ with $\delta = \delta(\varepsilon) > 0$.*

b. *$G_\varepsilon(t, t + 0; \alpha)$ is a trace-class operator and $\operatorname{tr} BG_\varepsilon(t, t + 0; \alpha)$ is integrable with respect to t and α such that for the section determinants the following integral formula holds:*

$$\det P(0)P(1)P(0) = \lim_{\eta \to +0} \lim_{\varepsilon \to +0} \exp \left[-2 \operatorname{Re} \int_0^1 \int_{-\infty}^{0} e^{\eta t} \operatorname{tr} BG_\varepsilon(t, t + 0; \alpha) \, dt \, d\alpha \right]. \quad (5.2)$$

The requirement that there is an integration path Γ with $\Gamma \subset \rho(A(\alpha))$ is satisfied for example if $\|B\|$ is small enough (see Theorem 2.11). The requirement that $N \geq 1$ makes sense because otherwise $P(0)$ is the null operator.

As explained in the introduction this theorem may be considered an adiabatic theorem. We briefly recall the classical adiabatic theorem (see the introduction and [2]). The spectral projections of a family of self-adjoint operators can be asymptotically transformed into each other, i.e., for $\eta \to 0$ we have:

$$P(s) = U_\eta(s)P(0)U_\eta^*(s) + O(\eta).$$

The operators U_η satisfy a Schrödinger equation:

$$i\eta \frac{d}{ds}U_\eta(s) = H(s)U_\eta(s),\ U_\eta(0) = \mathbb{1}.$$

The validity of this theorem requires a gap in the spectrum of all $H(s)$. This corresponds to the gap condition 1. in Theorem 5.1. In some way this is related to the parameter η. This is the reason why we chose the same notation as in the classical theorem. Besides we have the gap condition 4. associated with the parameter ε. This refers to the numerical range rather than the spectrum. In general the two conditions 1. and 4. cannot be deduced from each other. From 4. we deduce together with the dichotomy criterion 3.5 that the operator

$$A_\varepsilon(\alpha) := A(\alpha) + i\varepsilon(2P(0) - \mathbb{1})$$

has no real spectral values for $\varepsilon \neq 0$. This is essential for the proof. At this point we also see why the criterion 3.3 of Baskakov cannot be applied because the imaginary part of $A_\varepsilon(\alpha)$ becomes small as $\varepsilon \to 0$.

We divide the proof of Theorem 5.1 into four steps.

Step 1: As in Section 4 we start with the integral formula for finite dimensional determinants but here we use the version for orthogonal projections from Lemma 2.9:

$$\det P(0)P(1)P(0) = \det P(0)P(0)P(0) \exp\left[2 \operatorname{Re} \int_0^1 \operatorname{tr} P^+(\alpha)(\mathbb{1} - P(\alpha))P'(\alpha)P(\alpha)\, d\alpha\right]. \tag{5.3}$$

Here $P^+(\alpha)$ is the pseudo inverse of $P(\alpha)$ with respect to $P(0)$ as in Theorem 2.8.

The derivative $P'(\alpha)$ can be calculated via Theorem 2.16:

$$P'(\alpha) = \frac{1}{2\pi i} \int_\Gamma (z\mathbb{1} - A(\alpha))^{-1} B (z\mathbb{1} - A(\alpha))^{-1}\, dz. \tag{5.4}$$

Step 2: The resolvent is essentially the Fourier transform of the unitary group. A purely formal application of the Parseval identity of the Fourier transform to the integral in (5.4) would yield a link to the Schrödinger equation. This can be made precise by shifting the spectrum of $A(\alpha)$ into the complex plane. In order to do this we add two imaginary parts:

$$A_\varepsilon(\alpha) := A(\alpha) + i\varepsilon(2P(0) - \mathbb{1}),\ \varepsilon \in \mathbb{R}, \tag{5.5}$$

$$A_{\eta,\varepsilon}(\alpha) := A_\varepsilon(\alpha) + \frac{i\eta}{2}(2P_\varepsilon(\alpha) - \mathbb{1}),\ \eta \in \mathbb{R}, \tag{5.6}$$

$P_\varepsilon(\alpha)$ being the Riesz projection of $A_\varepsilon(\alpha)$.

Step 3: For $\eta \neq 0$ the operator $A_{\eta,\varepsilon}(\alpha)$ possesses no real spectral values. Therefore we can use the Laplace transform to yield the group $U_\varepsilon(t,\alpha)$. After dropping a phase factor we arrive at the formula:

$$\begin{aligned} &\det P(0)P(1)P(0) \\ &= \lim_{\eta\to+0} \lim_{\varepsilon\to 0} \exp\left[-2 \operatorname{Re} i \int_0^1 \int_{-\infty}^0 e^{\eta t} \operatorname{tr} B U_\varepsilon(t,\alpha) P_\varepsilon^+(\alpha) P_\varepsilon(\alpha) U_\varepsilon(-t,\alpha)\, dt\, d\alpha\right]. \end{aligned} \tag{5.7}$$

The factor $e^{\eta t}$ is a consequence of the perturbation in (5.6).

Step 4: Analogously to Section 4 we describe the exponent in (5.7) in terms not depending on U_ε. Essentially we define $G_\varepsilon(t,t;\alpha)$ as the exponent in (5.7) and show with the aid of the integral equation in Theorem 2.42 that for $\varepsilon > 0$, $G_\varepsilon(t,t';\alpha)$ satisfies the Wiener-Hopf equation (5.1). The factor $e^{-\varepsilon|t-\tau|}$ in (5.1) is due to the perturbation in (5.5).

The proof differs to that of Section 4.2 by the additional steps 2 and 3. The shift of the spectrum is necessary for applying the Laplace transform. This makes it possible to relate the resolvent to the group leading to a formula analogous to that of Section 4.2.

5.2 Proof of the integral formula

We state some properties which can be derived easily from the assumptions.

Lemma 5.2. *For all $\alpha \in [0,1]$ holds:*

1. *$A(\alpha)$ is self-adjoint on $D(A_0)$ and $P(\alpha)$ is an orthogonal projection.*
2. *The range of $P(\alpha)$ is finite dimensional with $\dim \operatorname{ran} P(\alpha) = N$.*
3. *The spectral set of $A(\alpha)$ with respect to Γ contains exactly N eigenvalues (up to multiplicity) and no other spectral values.*

Proof. 1. $A(\alpha)$ is self-adjoint because the sum of a self-adjoint operator and a bounded symmetric operator is again self-adjoint (see [13]). $P(\alpha)$ is the Riesz projection of a self-adjoint operator and hence an orthogonal projection (Theorem 2.15).

2. By Theorem 2.16 the $P(\alpha)$ are continuously differentiable and thus continuous in α. Therefore it follows from Lemma 2.10 that $\dim \operatorname{ran} P(\alpha) = \dim \operatorname{ran} P(0) = N$.

3. Follows from 2. and Corollary 2.14. □

5.2.1 Step 1: The finite dimensional determinant formula

The first step just consists of a simple application of Lemma 2.9 to the family of Riesz projections of $A(\alpha)$.

Theorem 5.3. *For the family of Riesz projections $P(\alpha)$ of $A(\alpha)$ there holds:*

$$\det P(0)P(1)P(0) = \exp\left[2\,\mathrm{Re}\int_0^1 \operatorname{tr} P^+(\alpha)(\mathbb{1} - P(\alpha))P'(\alpha)P(\alpha)\,d\alpha\right]. \tag{5.8}$$

Here $P^+(\alpha)$ is the pseudo inverse of $P(\alpha)$ with respect to $P(0)$ and $P'(\alpha)$ is given by a Friedrichs integral:

$$P'(\alpha) = \frac{1}{2\pi i}\int_\Gamma (z\mathbb{1} - A(\alpha))^{-1} B (z\mathbb{1} - A(\alpha))^{-1}\,dz. \tag{5.9}$$

Proof. The $P(\alpha)$ are continuously differentiable with respect to α in the operator norm by Theorem 2.16 and the derivative is given by the Friedrichs integral (5.9). We apply Lemma 2.9 to the finite dimensional operator $P(0)P(\alpha)P(0) : \operatorname{ran} P(0) \to \operatorname{ran} P(0)$ and note $\det P(0)P(0)P(0) = 1$ (because of $N \geq 1$). Then the theorem follows. □

5.2.2 Step 2: Shifting the spectrum

We add an imaginary part to $A(\alpha)$:

$$A_\varepsilon(\alpha) := A(\alpha) + i\varepsilon J,\ J := 2P(0) - \mathbb{1},\ \varepsilon \in \mathbb{R}. \tag{5.10}$$

This kind of operators was studied in Chapter 3. We note:

$$J^2 = 4P(0)^2 - 4P(0) + \mathbb{1} = \mathbb{1},\ J^* = J,\ \|J\| = 1. \tag{5.11}$$

Though $A_\varepsilon(\alpha)$ is not symmetric it has nice properties.

Lemma 5.4. *Let $A_\varepsilon(\alpha)$, $\alpha \in [0,1]$, be as in (5.10). Then for all $\varepsilon \in \mathbb{R}$, $A_\varepsilon(\alpha)$ is a densely defined and closed operator and for all $\alpha \in [0,1]$ we have:*

1. *$\sigma(A_\varepsilon(\alpha)) \subset \{z \in \mathbb{C} \,|\, |\operatorname{Im} z| \leq |\varepsilon|\}$ for all $\varepsilon \in \mathbb{R}$. The resolvent can be estimated:*

$$\|(z\mathbb{1} - A_\varepsilon(\alpha))^{-1}\| \leq \frac{1}{|\operatorname{Im} z| - |\varepsilon|}, \quad |\operatorname{Im} z| > |\varepsilon|. \tag{5.12}$$

2. *We choose $\varepsilon_0 > 0$ such that $\varepsilon_0 m < 1$ where*

$$m := \max_{\substack{z\in\Gamma \\ \alpha\in[0,1]}} \|(z\mathbb{1} - A(\alpha))^{-1}\| < \infty \tag{5.13}$$

is well defined.[1] *Then $\Gamma \subset \rho(A_\varepsilon(\alpha))$ for all $|\varepsilon| \leq \varepsilon_0$ and we have for $z \in \Gamma$:*

$$\|(z\mathbb{1} - A_\varepsilon(\alpha))^{-1}\| \leq \frac{m}{1 - \varepsilon_0 m} =: m_1. \tag{5.14}$$

Proof. Because J is bounded by (5.11) $A_\varepsilon(\alpha)$ is defined on $D(A(\alpha)) = D(A_0)$. The sum of a closed operator and a bounded operator is again bounded.

1. We examine the resolvent. For $z \in \rho(A(\alpha))$ we have:

$$z\mathbb{1} - A(\alpha) - i\varepsilon J = (z\mathbb{1} - A(\alpha))(\mathbb{1} - i\varepsilon(z\mathbb{1} - A(\alpha))^{-1}J). \tag{5.15}$$

Let $\operatorname{Im} z \neq 0$. From the inequality (3.1) for the resolvent together with $\|J\| = 1$ (see (5.11)) it follows:

$$\|i\varepsilon(z\mathbb{1} - A(\alpha))^{-1}J\| \leq |\varepsilon|\|(z\mathbb{1} - A(\alpha))^{-1}\|\|J\| \leq |\varepsilon|\frac{1}{|\operatorname{Im} z|}.$$

For $|\operatorname{Im} z| > |\varepsilon|$ the Neumann series yields the existence of $(z\mathbb{1} - A(\alpha) - i\varepsilon J)^{-1}$ and the inequality for the resolvent. This gives the statement about the location of the spectrum.

2. By assumption we have $\Gamma \subset \rho(A(\alpha))$ for all $\alpha \in [0,1]$. Therefore (5.15) holds also for $z \in \Gamma$. The function $(z,\alpha) \mapsto \|(z\mathbb{1} - A(\alpha))^{-1}\|$ is continuous by Theorem 2.11 and thus takes its maximum $0 \leq m < \infty$ on the compact set $\Gamma \times [0,1]$. Then we have for all $|\varepsilon| \leq \varepsilon_0$ and all $\alpha \in [0,1]$ by assumption:

$$\|i\varepsilon(z\mathbb{1} - A(\alpha))^{-1}J\| \leq m|\varepsilon|\|J\| \leq m\varepsilon_0 < 1.$$

Now the Neumann series yields $z \in \rho(A_\varepsilon(\alpha))$ and thus $\Gamma \subset \rho(A_\varepsilon(\alpha))$. Furthermore we have:

$$\|(z\mathbb{1} - A_\varepsilon(\alpha))^{-1}\| \leq \frac{m}{1 - \varepsilon_0 m}.$$

This completes the proof. □

$A_\varepsilon(\alpha)$ possesses Riesz projections and generates a group of bounded operators.

Lemma 5.5. *Let $A_\varepsilon(\alpha)$ be as in (5.10). Let ε_0 be chosen as in Lemma 5.4. Then for all $\alpha \in [0,1]$ and all $|\varepsilon| \leq \varepsilon_0$:*

[1] It may be possible to avoid the restriction to sufficiently small ε by deforming the integration path. But because we have to perform the limit $\varepsilon \to 0$ anyway it seems to be reasonable to drop this generalisation that only would necessitate more technical calculations.

1. *The Riesz projections*

$$P_\varepsilon(\alpha) = \frac{1}{2\pi i}\int_\Gamma (z\mathbb{1} - A_\varepsilon(\alpha))^{-1}\,dz$$

exist and can be estimated with m_1 *from (5.14):*

$$\|P_\varepsilon(\alpha)\| \le \frac{1}{2\pi}|\Gamma| m_1 =: C_P. \tag{5.16}$$

2. *The* $P_\varepsilon(\alpha)$ *are continuous in* ε *uniformly for all* $\alpha \in [0,1]$. *Especially we have* $\dim \operatorname{ran} P_\varepsilon(\alpha) = N$ *and the spectral set of* $A_\varepsilon(\alpha)$ *with respect to* Γ *contains exactly* N *eigenvalues and no other spectral values.*

Proof. 1. By Lemma 5.4 $\Gamma \subset \rho(A_\varepsilon(\alpha))$ showing the existence of the Riesz projections. (5.16) is an immediate consequence of the estimate (5.14) for the resolvent in Lemma 5.4 and the standard estimate for contour integrals.

2. Let $|\varepsilon_1|,\ |\varepsilon_2| \le \varepsilon_0$. With the aid of the estimate (5.14) for the resolvent we get:

$$\begin{aligned}\|(z\mathbb{1} - A_{\varepsilon_2}(\alpha))^{-1} - (z\mathbb{1} - A_{\varepsilon_1}(\alpha))^{-1}\| &= \|i(\varepsilon_2 - \varepsilon_1)(z\mathbb{1} - A_{\varepsilon_2}(\alpha))^{-1}J(z\mathbb{1} - A_{\varepsilon_1}(\alpha))^{-1}\| \\ &\le |\varepsilon_2 - \varepsilon_1| \|(z\mathbb{1} - A_{\varepsilon_2}(\alpha))^{-1}\| \|J\| \|(z\mathbb{1} - A_{\varepsilon_1}(\alpha))^{-1}\| \\ &\le |\varepsilon_2 - \varepsilon_1| m_1^2.\end{aligned}$$

Because this is correct for all $z \in \Gamma$ and all $\alpha \in [0,1]$ the continuity of the Riesz projections in ε uniformly in α follows. The dimension of the range is a consequence of Lemma 2.10 and the fact $\dim \operatorname{ran} P(\alpha) = N$. This also yields via Corollary 2.14 the statement about the spectral set. □

Lemma 5.6. *Let* $A_\varepsilon(\alpha)$ *as in (5.10). Then for all* $\alpha \in [0,1]$*:*

1. *For all* $\varepsilon \in \mathbb{R}$, $A_\varepsilon(\alpha)$ *generates a group* $U_\varepsilon(t,\alpha)$ *with:*

$$\|U_\varepsilon(t,\alpha)\| \le e^{|\varepsilon t|},\ t \in \mathbb{R}. \tag{5.17}$$

2. *Let* $\varepsilon > 0$. *Then there is a* $C = C(\varepsilon) \ge 0$ *and a* $\delta = \delta(\varepsilon) > 0$ *such that*

$$\|U_\varepsilon(t,\alpha)P_\varepsilon(\alpha)\| \le Ce^{\delta t},\ t \le 0. \tag{5.18}$$

Proof. 1. By Lemma 5.4 $-iA_\varepsilon(\alpha)$ satisfies the assumptions of Theorem 2.22 with $M = 1$ and $\omega = |\varepsilon|$ and therefore generates a group with the stated property.

2. We consider the group $\tilde{U}_\varepsilon(t,\alpha)$ generated by $-iA_\varepsilon(\alpha)P_\varepsilon(\alpha)$. Because $P_\varepsilon(\alpha)$ is the Riesz projection of $A_\varepsilon(\alpha)$ we have $P_\varepsilon(\alpha)\varphi \in D(A_\varepsilon(\alpha))$ for all $\varphi \in \mathcal{H}$ and $A_\varepsilon(\alpha)P_\epsilon(\alpha)\varphi = P_\epsilon(\alpha)A_\varepsilon(\alpha)\varphi$ for all $\varphi \in D(A_\varepsilon(\alpha))$. By Theorem 2.36 $P_\varepsilon(\alpha)$ commutes with $U_\varepsilon(t,\alpha)$. Thus the assumptions of Lemma 2.38 are satisfied and we conclude $\tilde{U}_\varepsilon(t,\alpha)P_\varepsilon(\alpha) = U_\varepsilon(t,\alpha)P_\varepsilon(\alpha)$.

In order to prove the estimate (5.18) it therefore suffices to consider the finite dimensional operator $\tilde{U}_\varepsilon(t,\alpha)P_\varepsilon(\alpha) : \operatorname{ran} P_\varepsilon(\alpha) \to \operatorname{ran} P_\varepsilon(\alpha)$. It is determined by the N-dimensional system of differential equations:

$$i\frac{d}{dt}\tilde{U}_\varepsilon(t,\alpha)P_\varepsilon(\alpha) = A_\varepsilon(\alpha)P_\varepsilon(\alpha)\tilde{U}_\varepsilon(t,\alpha)P_\varepsilon(\alpha).$$

We shall show that $A_\varepsilon(\alpha)P_\varepsilon(\alpha)$ has only eigenvalues with positive imaginary part. Then Corollary 2.29 gives the desired estimate.

The spectrum $\tilde{\sigma}$ of $A_\varepsilon(\alpha)P_\varepsilon(\alpha) : \operatorname{ran} P_\varepsilon(\alpha) \to \operatorname{ran} P_\varepsilon(\alpha)$ coincides with the spectral set of $A_\varepsilon(\alpha)$ with respect to Γ by Theorem 2.13. Therefore $\tilde{\sigma}$ contains only a finite number of eigenvalues (Lemma 5.5). Because of 4 in Theorem 5.1 $A(\alpha)$ satisfies the dichotomy criterion 3.5. Thus $A_\varepsilon(\alpha)$ has no real spectral values. We now use a continuity argument to show that all eigenvalues of $A_\varepsilon(\alpha)P_\varepsilon(\alpha)$ have a positive imaginary part. To do this we take into account that $P_\varepsilon(0) = P(0)$ for ε sufficiently small (Corollary 2.17). Then $(A_0 + i\varepsilon J)P_\varepsilon(0) = (A_0 + i\varepsilon J)P(0) = (A_0 + i\varepsilon P(0))P(0) : \operatorname{ran} P(0) \to \operatorname{ran} P(0)$ has only eigenvalues with positive imaginary part. Because a finite number of eigenvalues depend continuously on α ([10], IV.3.5) all eigenvalues of $A_\varepsilon(\alpha)P_\varepsilon(\alpha)$ must have a positive imaginary part. $\square$

We add another imaginary part in order to ensure that there are no spectral values in a strip around the real axis:

$$A_{\eta,\varepsilon}(\alpha) = A_\varepsilon(\alpha) + \frac{i\eta}{2} J_\varepsilon(\alpha),\ J_\varepsilon(\alpha) := 2P_\varepsilon(\alpha) - \mathbb{1},\ \eta \in \mathbb{R}. \tag{5.19}$$

The factor $1/2$ leads to $e^{\eta t}$ in the integral formula (5.2) rather than $e^{2\eta t}$ (see the introduction for the connection with the adiabatic theorem). From (5.16) in Lemma 5.5 it follows:

$$\|J_\varepsilon(\alpha)\| \le 2\|P_\varepsilon(\alpha)\| + 1 \le 2C_P + 1 =: C_J. \tag{5.20}$$

$A_{\eta,\varepsilon}(\alpha)$ possesses Riesz projections and generates a group, too. But we do not need the Riesz projections and the group can be expressed by $U_\varepsilon(t, \alpha)$.

Lemma 5.7. *Let the operators $A_{\eta,\varepsilon}(\alpha)$ be as in (5.19). Then for all $\alpha \in [0, 1]$ we have:*

1. *$A_{\eta,\varepsilon}(\alpha) : D(A_0) \to \mathcal{H}$ is a closed operator. Pick $\eta_0 > 0$ such that $\frac{\eta_0}{2} m_1 C_J < 1$. Then $\Gamma \subset \rho(A_{\eta,\varepsilon}(\alpha))$ for all $|\eta| \le \eta_0$ and the resolvent can be estimated by:*

$$\|(z\mathbb{1} - A_{\eta,\varepsilon}(\alpha))^{-1}\| \le \frac{m_1}{1 - \frac{\eta_0}{2} m_1 C_J} =: m_2. \tag{5.21}$$

2. *$-iA_{\eta,\varepsilon}(\alpha)$ generates a group of bounded operators $U_{\eta,\varepsilon}(t, \alpha)$ with:*

$$U_{\eta,\varepsilon}(t, \alpha) = \tilde{U}_{\eta,\varepsilon}(t, \alpha)U_\varepsilon(t, \alpha) = U_\varepsilon(t, \alpha)\tilde{U}_{\eta,\varepsilon}(t, \alpha). \tag{5.22}$$

Here $\tilde{U}_{\eta,\varepsilon}(t, \alpha)$ is the group generated by $\frac{\eta}{2} J_\varepsilon(\alpha)$. In particular:

$$U_{\eta,\varepsilon}(t, \alpha)P_\varepsilon(\alpha) = e^{\frac{\eta t}{2}} U_\varepsilon(t, \alpha)P_\varepsilon(\alpha), \tag{5.23}$$

$$U_{\eta,\varepsilon}(t, \alpha)(\mathbb{1} - P_\varepsilon(\alpha)) = e^{-\frac{\eta t}{2}} U_\varepsilon(t, \alpha)(\mathbb{1} - P_\varepsilon(\alpha)). \tag{5.24}$$

Proof. 1. Since $J_\varepsilon(\alpha)$ is bounded $A_{\eta,\varepsilon}(\alpha)$ is defined on $D(A_0)$ and as the sum of the closed operator $A_\varepsilon(\alpha)$ and the bounded operator $J_\varepsilon(\alpha)$ it is again a closed operator.

Pick $z \in \Gamma$. By Lemma 5.4 we know $\Gamma \subset \rho(A_\varepsilon(\alpha))$. Therefore:

$$z\mathbb{1} - A_{\eta,\varepsilon}(\alpha) = z\mathbb{1} - A_\varepsilon(\alpha) - \frac{i\eta}{2} J_\varepsilon(\alpha) = (z\mathbb{1} - A_\varepsilon(\alpha))(\mathbb{1} - \frac{i\eta}{2}(z\mathbb{1} - A_\varepsilon(\alpha))^{-1} J_\varepsilon(\alpha)).$$

We use the estimates (5.14) and (5.20) and conclude:

$$\|(z\mathbb{1} - A_\varepsilon(\alpha))^{-1} J_\varepsilon(\alpha)\| \le \|(z\mathbb{1} - A_\varepsilon(\alpha))^{-1}\| \|J_\varepsilon(\alpha)\| \le m_1 C_J$$

The assumption on η_0 is chosen in such a way that the Neumann series gives the existence of the resolvent and the estimate (5.21).

2. $-iA_{\eta,\varepsilon}(\alpha)$ satisfies the conditions of Theorem 2.31 because $-iA_\varepsilon(\alpha)$ is the generator of a group and $J_\varepsilon(\alpha)$ is bounded (see (5.20)). Hence $-iA_{\eta,\varepsilon}(\alpha)$ generates the group $U_{\eta,\varepsilon}(t,\alpha)$. By Theorem 2.13 $A_\varepsilon(\alpha)$ commutes with $J_\varepsilon(\alpha)$. Therefore Theorem 2.37 yields the formula (5.22).

We have $J_\varepsilon(\alpha)^2 = \mathbb{1}$ whence the formulas (5.23) and (5.24) follow immediately from Lemma 2.39. □

In order to use the preceding results we prove some convergence results.

Lemma 5.8. *The following convergence statements hold with respect to the operator norm and uniformly for $\alpha \in [0,1]$:*

1. $\lim_{\varepsilon\to 0} P_\varepsilon(\alpha) = P(\alpha)$.
2. *There is an $\tilde{\varepsilon}_0 > 0$ such that the pseudo inverse $P_\varepsilon^+(\alpha)$ of $P_\varepsilon(\alpha)$ with respect to $P(0)$ exists for all $|\varepsilon| \le \tilde{\varepsilon}_0$ and all $\alpha \in [0,1]$. Furthermore $P_\varepsilon^+(\alpha) \to P^+(\alpha)$ for $\varepsilon \to 0$.*
3. *For $|\varepsilon| \le \varepsilon_0$ and $|\eta| \le \eta_0$ let $\Phi_{\eta,\varepsilon}(\alpha)$ be defined as Friedrichs integral:*

$$\Phi_{\eta,\varepsilon}(\alpha) := \frac{1}{2\pi i} \int_\Gamma (z\mathbb{1} - A_{\eta,\varepsilon}(\alpha))^{-1} B (z\mathbb{1} - A_{\eta,\varepsilon}(\alpha))^{-1}\, dz.$$

Then $\Phi_{\eta,\varepsilon}(\alpha) \to P'(\alpha)$ for $\eta,\varepsilon \to 0$ independently of the order of the limits.

Proof. 1. The uniform convergence of the Riesz projections is a trivial consequence of the continuity in Lemma 5.5.

2. By assumption $P^+(\alpha)$ exists for all $\alpha \in [0,1]$. We have:

$$\begin{aligned} P(0)P_\varepsilon(\alpha)P(0) &= P(0)P(\alpha)P(0) + P(0)P_\varepsilon(\alpha)P(0) - P(0)P(\alpha)P(0) \\ &= P(0)P(\alpha)P(0)[\mathbb{1} + P^+(\alpha)(P(0)P_\varepsilon(\alpha)P(0) - P(0)P(\alpha)P(0))]. \end{aligned} \tag{5.25}$$

$P^+(\alpha)$ depends continuously on α because $P(\alpha)$ does (Theorems 2.16 and 2.5). Therefore the continuous function $\alpha \mapsto \|P^+(\alpha)\|$ takes its maximum on the compact set $[0,1]$:

$$\tilde{m} := \max_{\alpha\in[0,1]} \|P^+(\alpha)\| < \infty.$$

With this we can estimate in (5.25):

$$\|P^+(\alpha)(P(0)P_\varepsilon(\alpha)P(0) - P(0)P(\alpha)P(0))\| \le \tilde{m}\|P_\varepsilon(\alpha) - P(\alpha)\|.$$

Pick $0 < \tilde{M} < 1$. By part 1 there is an $\tilde{\varepsilon}_0 > 0$ such that for all ε with $|\varepsilon| \le \tilde{\varepsilon}_0$ and all $\alpha \in [0,1]$ we have:

$$\tilde{m}\|P_\varepsilon(\alpha) - P(\alpha)\| \le \tilde{M} < 1.$$

Therefore the Neumann series shows that $P_\varepsilon^+(\alpha)$ exists with:

$$\|P_\varepsilon^+(\alpha)\| \le \frac{\tilde m}{1-\tilde M} =: \tilde C. \tag{5.26}$$

We show convergence:

$$\|P_\varepsilon^+(\alpha) - P^+(\alpha)\| \le \|P_\varepsilon^+(\alpha)\|\|P(\alpha)-P_\varepsilon(\alpha)\|\|P(\alpha)\| \le \frac{\tilde m^2}{1-\tilde M}\|P_\varepsilon(\alpha)-P(\alpha)\|.$$

Together with 1. this yields the statement.

3. The Friedrichs integral is well defined by Lemma 5.7. We have:

$$\begin{aligned}
&(z\mathbb{1} - A_{\eta,\varepsilon}(\alpha))^{-1}B(z\mathbb{1} - A_{\eta,\varepsilon}(\alpha))^{-1} - (z\mathbb{1} - A(\alpha))^{-1}B(z\mathbb{1} - A(\alpha))^{-1}\\
&= (z\mathbb{1} - A_{\eta,\varepsilon}(\alpha))^{-1}B(z\mathbb{1} - A_{\eta,\varepsilon}(\alpha))^{-1}(i\varepsilon J + \frac{i\eta}{2}J_\varepsilon(\alpha))(z\mathbb{1} - A(\alpha))^{-1}\\
&\quad + (z\mathbb{1} - A_{\eta,\varepsilon}(\alpha))^{-1}(i\varepsilon J + \frac{i\eta}{2}J_\varepsilon(\alpha))(z\mathbb{1} - A(\alpha))^{-1}B(z\mathbb{1} - A(\alpha))^{-1}
\end{aligned}$$

We estimate:

$$\begin{aligned}
&\|(z\mathbb{1} - A_{\eta,\varepsilon}(\alpha))^{-1}B(z\mathbb{1} - A_{\eta,\varepsilon}(\alpha))^{-1} - (z\mathbb{1} - A(\alpha))^{-1}B(z\mathbb{1} - A(\alpha))^{-1}\|\\
&\qquad\qquad \le m_2\|B\|m_2(|\varepsilon| + \frac{|\eta|}{2}C_J)m + m_2 m(|\varepsilon| + \frac{|\eta|}{2}C_J)\|B\|m.
\end{aligned}$$

Since this holds independently of $z \subset \Gamma$ and $\alpha \in [0,1]$ the uniform convergence of the Friedrichs integrals follows. Moreover we conclude that the order of the limits is not essential. □

We state the result of the second step.

Theorem 5.9. *Let $P_\varepsilon(\alpha)$ be the pseudo inverse of $P_\varepsilon(\alpha)$ with respect to $P(0)$ and $\Phi_{\eta,\varepsilon}(\alpha)$ the Friedrichs integral as in Lemma 5.8. Then we have:*

$$\det P(0)P(1)P(0) = \lim_{\eta\to 0}\exp\left[2\,\mathrm{Re}\int_0^1 \lim_{\varepsilon\to 0}\mathrm{tr}\, P_\varepsilon^+(\alpha)(\mathbb{1} - P_\varepsilon(\alpha))\Phi_{\eta,\varepsilon}(\alpha)P_\varepsilon(\alpha)\, d\alpha\right].$$

Proof. By Lemma 5.8 all quantities converge in the operator norm for $\eta \to 0$ and $\varepsilon \to 0$ uniformly for $\alpha \in [0,1]$ toward the corresponding expressions in (5.8) in Theorem 5.3. Since $P_\varepsilon^+(\alpha) = P(0)P_\varepsilon^+(\alpha)$ and $\dim\mathrm{ran}\, P(0) = N$ this holds also for the trace by Lemma 2.3. Therefore we can interchange the η-limit with the α-integration. □

We did not interchange the ε-limit with view to further calculations.

5.2.3 Step 3: Application of the Laplace transform

We replace the path Γ by $\tilde\Gamma$ having the property $\mathrm{Im}\, z \ge 0$ for $z \in \Gamma$. This is necessary for applying the Laplace transform.

Lemma 5.10. *Let $\frac{\eta}{2} > |\varepsilon|$. Then there is a path $\tilde{\Gamma} \subset \rho(A_{\eta,\varepsilon})$ such that the spectral sets of $A_{\eta,\varepsilon}(\alpha)$ with respect to $\tilde{\Gamma}$ and Γ coincide, respectively, and that* $\operatorname{Im} z \geq 0$ *for* $z \in \tilde{\Gamma}$. *We have:*

$$\int_\Gamma (z\mathbb{1} - A_{\eta,\varepsilon}(\alpha))^{-1} B (z\mathbb{1} - A_{\eta,\varepsilon}(\alpha))^{-1}\, dz = \int_{\tilde{\Gamma}} (z\mathbb{1} - A_{\eta,\varepsilon}(\alpha))^{-1} B (z\mathbb{1} - A_{\eta,\varepsilon}(\alpha))^{-1}\, dz.$$

Proof. The spectrum $\tilde{\sigma}$ of $A_{\eta,\varepsilon}(\alpha)P_\varepsilon(\alpha) : \operatorname{ran} P_\varepsilon(\alpha) \to \operatorname{ran} P_\varepsilon(\alpha)$ contains no real eigenvalues because we have:

$$A_{\eta,\varepsilon}(\alpha)P_\varepsilon(\alpha) = A_\varepsilon(\alpha)P_\varepsilon(\alpha) + \frac{i\eta}{2}P_\varepsilon(\alpha).$$

Since $P_\varepsilon(\alpha)$ is the Riesz projection of $A_\varepsilon(\alpha)$ by Theorem 2.13 the spectrum of $A_\varepsilon(\alpha)P_\varepsilon(\alpha) : \operatorname{ran} P_\varepsilon(\alpha) \to \operatorname{ran} P_\varepsilon(\alpha)$ coincides with the spectral set of $A_\varepsilon(\alpha)$ with respect to Γ. The latter consists only of a finite number of eigenvalues. Since $P_\varepsilon(\alpha)$ is the identity on $\operatorname{ran} P_\varepsilon(\alpha)$ the assumption gives:

$$\operatorname{Im} \mu \geq \frac{\eta}{2} - |\varepsilon| > 0,\ \mu \in \sigma_1(A_{\eta,\varepsilon}(\alpha)).$$

Thus we can choose $\tilde{\Gamma} \subset \rho(A_{\eta,\varepsilon}(\alpha))$ in such a way that $\operatorname{Im} z \geq 0$ for $z \in \tilde{\Gamma}$ and the spectral set does not change during the transition from Γ to $\tilde{\Gamma}$. The Cauchy integral theorem then shows the equality of the Friedrichs integrals. $\square$

The parameters ε and η of the preceding step allow the application of Theorems 2.27 and 2.30 concerning the relation between resolvent and corresponding group.

Lemma 5.11. *Let ε_0 be as in Lemma 5.4 and η_0 as in Lemma 5.7. Let $|\varepsilon| \leq \varepsilon_0$ and $|\eta| \leq \eta_0$ and $\eta/2 > |\varepsilon|$. Then for all $\varphi \in \mathcal{H}$ the formula holds:*

$$(\mathbb{1} - P_\varepsilon(\alpha))\Phi_{\eta,\varepsilon}(\alpha)P_\varepsilon(\alpha)\varphi = -i\int_{-\infty}^0 e^{\eta t} U_\varepsilon(-t,\alpha)(\mathbb{1} - P_\varepsilon(\alpha)) B U_\varepsilon(t,\alpha) P_\varepsilon(\alpha)\varphi\, dt. \quad (5.27)$$

Proof. From Lemma 5.7 we get the estimate for $t \in \mathbb{R}$:

$$\|U_{\eta,\varepsilon}(-t,\alpha)(\mathbb{1} - P_\varepsilon(\alpha))\| \leq e^{\frac{\eta t}{2}} \|U_\varepsilon(-t,\alpha)\| \|\mathbb{1} - P_\varepsilon(\alpha)\| \leq e^{\frac{\eta t}{2}} e^{|\varepsilon t|} \|\mathbb{1} - P_\varepsilon(\alpha)\|. \quad (5.28)$$

For $\eta/2 > |\varepsilon|$ and $\operatorname{Im} z \geq 0$ Theorem 2.30 yields the formula:

$$(z\mathbb{1} - A_{\eta,\varepsilon}(\alpha))^{-1}(\mathbb{1} - P_\varepsilon(\alpha))\varphi = -i\int_{-\infty}^0 e^{-izt} U_{\eta,\varepsilon}(-t,\alpha)(\mathbb{1} - P_\varepsilon(\alpha))\varphi\, dt.$$

We insert this into the Friedrichs integral in Lemma 5.10 with the path $\tilde{\Gamma}$ where we use that $\mathbb{1} - P_\varepsilon(\alpha)$ commutes with $(z\mathbb{1} - A_{\eta,\varepsilon}(\alpha))^{-1}$:

$$\begin{aligned}
&(\mathbb{1} - P_\varepsilon(\alpha))\Phi_{\eta,\varepsilon}(\alpha)P_\varepsilon(\alpha)\varphi \\
&= \frac{1}{2\pi i}\int_{\tilde{\Gamma}} (\mathbb{1} - P_\varepsilon(\alpha))(z\mathbb{1} - A_{\eta,\varepsilon}(\alpha))^{-1} B (z\mathbb{1} - A_{\eta,\varepsilon}(\alpha))^{-1} P_\varepsilon(\alpha)\varphi\, dz \\
&= -\frac{i}{2\pi i}\int_{\tilde{\Gamma}}\int_{-\infty}^0 e^{-izt} U_{\eta,\varepsilon}(-t,\alpha)(\mathbb{1} - P_\varepsilon(\alpha)) B (z\mathbb{1} - A_{\eta,\varepsilon}(\alpha))^{-1} P_\varepsilon(\alpha)\varphi\, dt\, dz \\
&= -\frac{i}{2\pi i}\int_{-\infty}^0 U_{\eta,\varepsilon}(-t,\alpha)(\mathbb{1} - P_\varepsilon(\alpha)) B \int_{\tilde{\Gamma}} e^{-izt}(z\mathbb{1} - A_{\eta,\varepsilon}(\alpha))^{-1} P_\varepsilon(\alpha)\varphi\, dz\, dt.
\end{aligned}$$

The interchange of the order of integration is justified by the estimate (5.28). The operator $A_{\eta,\varepsilon}(\alpha)P_\varepsilon(\alpha)$ is bounded by Theorem 2.13 and the resolvent of the operator restricted to $\operatorname{ran} P_\varepsilon(\alpha)$ equals the resolvent of the restriction. Hence the Dunford-Taylor integral in the last equation gives the group generated by $-iA_{\eta,\varepsilon}(\alpha)P_\varepsilon(\alpha)$ (see Theorem 2.27). With the aid of Lemma 2.38 we get:

$$\begin{aligned}(\mathbb{1} - P_\varepsilon(\alpha))\Phi_{\eta,\varepsilon}(\alpha)P_\varepsilon(\alpha)\varphi &= -i\int_{-\infty}^{0} U_{\eta,\varepsilon}(-t,\alpha)(\mathbb{1} - P_\varepsilon(\alpha))BU_{\eta,\varepsilon}(t,\alpha)P_\varepsilon(\alpha)\varphi\, dt \\ &= -i\int_{-\infty}^{0} e^{\eta t}U_\varepsilon(-t,\alpha)(\mathbb{1} - P_\varepsilon(\alpha))BU_\varepsilon(t,\alpha)P_\varepsilon(\alpha)\varphi\, dt.\end{aligned}$$

In the last step we used Lemma 5.7. $\square$

By the preceding Lemma 5.11 we have obtained a factor in the Friedrichs integral which ensures convergence and thus allows us to split up the integrand:

$$\begin{aligned}&U_\varepsilon(-t,\alpha)(\mathbb{1} - P_\varepsilon(\alpha))BU_\varepsilon(t,\alpha)P_\varepsilon(\alpha) \\ &= U_\varepsilon(-t,\alpha)BU_\varepsilon(t,\alpha)P_\varepsilon(\alpha) - U_\varepsilon(-t,\alpha)P_\varepsilon(\alpha)BU_\varepsilon(t,\alpha)P_\varepsilon(\alpha)\end{aligned}$$

The second summand corresponds to the term $P(\alpha)P'(\alpha)P(\alpha) = 0$ in Lemma 2.9. Therefore it is not surprising that it does not contribute to the real part.

Lemma 5.12. *The following convergence statements hold uniformly for all $\alpha \in [0,1]$:*

1. *Let $t \in \mathbb{R}$ be fixed. Then we have:*

$$\lim_{\varepsilon\to 0} \operatorname{Im}\operatorname{tr} P_\varepsilon^+(\alpha)U_\varepsilon(-t,\alpha)P_\varepsilon(\alpha)BU_\varepsilon(t,\alpha)P_\varepsilon(\alpha) = 0.$$

2. *For $\eta > 0$ we have:*

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{0} e^{\eta t}\operatorname{Im}\operatorname{tr} P_\varepsilon^+(\alpha)U_\varepsilon(-t,\alpha)P_\varepsilon(\alpha)BU_\varepsilon(t,\alpha)P_\varepsilon(\alpha)\, dt = 0.$$

Proof. 1. Since $P_\varepsilon^+(\alpha) = P(0)P_\varepsilon^+(\alpha)$ (see Theorem 2.8) and $P(0)$ has finite dimensional range it suffices by Theorem 2.3 to show convergence of the operators in the operator norm.

By Lemma 5.8 we have $P_\varepsilon(\alpha) \to P(\alpha)$ and $P_\varepsilon^+(\alpha) \to P^+(\alpha)$ for $\varepsilon \to 0$ uniformly in $\alpha \in [0,1]$. $U_\varepsilon(t,\alpha)$ converges for $t \in \mathbb{R}$ fixed and uniformly for $\alpha \in [0,1]$ to $U(t,\alpha)$ as is seen from Theorem 2.34:

$$\|U_\varepsilon(t,\alpha) - U(t,\alpha)\| \le e^{|\varepsilon|\|J\||t|} - 1 = e^{|\varepsilon||t|} - 1,$$

since $U(t,\alpha)$ is unitary. Thus we have uniformly in $\alpha \in [0,1]$:

$$\lim_{\varepsilon\to 0}\operatorname{Im}\operatorname{tr} P_\varepsilon^+(\alpha)U_\varepsilon(-t,\alpha)P_\varepsilon(\alpha)BU_\varepsilon(t,\alpha)P_\varepsilon(\alpha) = \operatorname{Im}\operatorname{tr} P^+(\alpha)U(-t,\alpha)P(\alpha)BU(t,\alpha)P(\alpha).$$

We note that B, $P(\alpha)$, and $P^+(\alpha)$ are self-adjoint and that $P(\alpha)$ commutes with $U(t,\alpha)$ (Theorem 2.36). It follows:

$$\begin{aligned}(\operatorname{tr} P^+(\alpha)U^*(t,\alpha)P(\alpha)BU(t,\alpha)P(\alpha))^* &= (\operatorname{tr} P^+(\alpha)U^*(t,\alpha)P(\alpha)BP(\alpha)U(t,\alpha))^* \\ &= \operatorname{tr}(P^+(\alpha)U^*(t,\alpha)P(\alpha)BP(\alpha)U(t,\alpha))^* \\ &= \operatorname{tr} U^*(t,\alpha)P(\alpha)BP(\alpha)U(t,\alpha)P^+(\alpha) \\ &= \operatorname{tr} P^+(\alpha)U^*(t,\alpha)P(\alpha)BP(\alpha)U(t,\alpha).\end{aligned}$$

Hence $\operatorname{tr} P^+(\alpha)U^*(t,\alpha)P(\alpha)BP(\alpha)U(t,\alpha) \in \mathbb{R}$. This yields 1.

2. The function

$$s_\varepsilon(t) := \sup_{\alpha\in[0,1]} |\operatorname{Im}\operatorname{tr} P_\varepsilon^+(\alpha)U_\varepsilon(-t,\alpha)P_\varepsilon(\alpha)BP_\varepsilon(\alpha)U_\varepsilon(t,\alpha)|$$

converges pointwise in $t \in \mathbb{R}$ with $\lim_{\varepsilon\to 0} s_\varepsilon(t) = 0$. Moreover we have for sufficiently small ε and all $\alpha \in [0,1]$:

$$|s_\varepsilon(t)| \le N\tilde{C}e^{2|\varepsilon t|}C_P^2\|B\|$$

(see Theorem 2.3, (5.26), (5.17), (5.16)). For $2|\varepsilon| < \eta$ the Lebesgue convergence theorem yields:

$$\lim_{\varepsilon\to 0}\int_{-\infty}^0 e^{\eta t}s_\varepsilon(t)\,dt = 0.$$

Since

$$|\int_{-\infty}^0 e^{\eta t}\operatorname{Im}\operatorname{tr} P_\varepsilon^+(\alpha)U_\varepsilon(-t,\alpha)P_\varepsilon(\alpha)BP_\varepsilon(\alpha)U_\varepsilon(t,\alpha)\,dt| \le \int_{-\infty}^0 e^{\eta t}s_\varepsilon(t)\,dt$$

the statement 2. follows. □

We can now return to Theorem 5.9 and thus get the main result of Step 3.

Theorem 5.13. *Let $A_\varepsilon(\alpha)$ be as in (5.10). Let $P_\varepsilon(\alpha)$ be the Riesz projection of $A_\varepsilon(\alpha)$, $P_\varepsilon^+(\alpha)$ the pseudo inverse of $P_\varepsilon(\alpha)$ with respect to $P(0)$, and $U_\varepsilon(t,\alpha)$ the group generated by $-iA_\varepsilon(\alpha)$. Then we have:*

$$\begin{aligned}&\det P(0)P(1)P(0)\\ &\quad= \lim_{\eta\to+0}\lim_{\varepsilon\to 0}\exp\left[-2\operatorname{Re} i\int_0^1\int_{-\infty}^0 e^{\eta t}\operatorname{tr} BU_\varepsilon(t,\alpha)P_\varepsilon(\alpha)P_\varepsilon^+(\alpha)U_\varepsilon(-t,\alpha)\,dt\,d\alpha\right].\end{aligned} \tag{5.29}$$

Proof. By Lemma 5.11 we have for $\varphi \in \mathcal{H}$:

$$P_\varepsilon^+(\alpha)(\mathbb{1}-P_\varepsilon(\alpha))\Phi_{\eta,\varepsilon}(\alpha)P_\varepsilon(\alpha)\varphi = -i\int_{-\infty}^0 e^{\eta t}P_\varepsilon^+(\alpha)U_\varepsilon(-t,\alpha)(\mathbb{1}-P_\varepsilon(\alpha))BU_\varepsilon(t,\alpha)P_\varepsilon(\alpha)\varphi\,dt.$$

Since $P_\varepsilon^+(\alpha) = P(0)P_\varepsilon(\alpha)$ and $\dim\operatorname{ran} P(0) = N$ this may be transferred to the trace:

$$\begin{aligned}I_\varepsilon(\alpha) &:= \operatorname{Re}\operatorname{tr} P_\varepsilon^+(\alpha)(\mathbb{1}-P_\varepsilon(\alpha))\Phi_{\eta,\varepsilon}(\alpha)P_\varepsilon(\alpha)\\ &= -\operatorname{Re} i\int_{-\infty}^0 e^{\eta t}\operatorname{tr} P_\varepsilon^+(\alpha)U_\varepsilon(-t,\alpha)(\mathbb{1}-P_\varepsilon(\alpha))BU_\varepsilon(t,\alpha)P_\varepsilon(\alpha)\,dt.\end{aligned} \tag{5.30}$$

Because of the estimate (5.17) for $U_\varepsilon(t,\alpha)$ the integrand in (5.30) can be split up for $\eta > 2|\varepsilon|$:

$$\begin{aligned}I_\varepsilon(\alpha) = &-\operatorname{Re} i\int_{-\infty}^0 e^{\eta t}\operatorname{tr} P_\varepsilon^+(\alpha)U_\varepsilon(-t,\alpha)BU_\varepsilon(t,\alpha)P_\varepsilon(\alpha)\,dt\\ &-\operatorname{Im}\int_{-\infty}^0 e^{\eta t}\operatorname{tr} P_\varepsilon^+(\alpha)U_\varepsilon(-t,\alpha)P_\varepsilon(\alpha)BU_\varepsilon(t,\alpha)P_\varepsilon(\alpha)\,dt\\ =&: I_{1,\varepsilon}(\alpha) + I_{2,\varepsilon}(\alpha).\end{aligned}$$

Here we have used that $U_\varepsilon(t,\alpha)$ commutes with $P_\varepsilon(\alpha)$. By Lemma 5.8 and Theorem 2.3 $\lim_{\varepsilon\to 0} I_\varepsilon(\alpha)$ exists uniformly for $\alpha\in[0,1]$. By Lemma 5.12 we have $\lim_{\varepsilon\to 0} I_{2,\varepsilon}(\alpha)=0$ uniformly for $\alpha\in[0,1]$. This gives:

$$\lim_{\varepsilon\to 0} I_\varepsilon(\alpha)=\lim_{\varepsilon\to 0} I_{1,\varepsilon}(\alpha)$$

uniformly for $\alpha\in[0,1]$. Starting from Theorem 5.13 we get

$$\begin{aligned}\det P(0)P(1)P(0) &= \lim_{\eta\to+0}\exp\left[2\int_0^1 \lim_{\varepsilon\to 0} I_\varepsilon(\alpha)\,d\alpha\right]\\ &= \lim_{\eta\to+0}\exp\left[2\int_0^1 \lim_{\varepsilon\to 0} I_{1,\varepsilon}(\alpha)\,d\alpha\right]\\ &= \lim_{\eta\to+0}\lim_{\varepsilon\to 0}\exp\left[2\int_0^1 I_{1,\varepsilon}(\alpha)\,d\alpha\right].\end{aligned}$$

The exponent can be rewritten:

$$I_{1,\varepsilon}(\alpha)=-\operatorname{Re} i\int_{-\infty}^0 e^{\eta t}\operatorname{tr} BU_\varepsilon(t,\alpha)P_\varepsilon(\alpha)P_\varepsilon^+(\alpha)U_\varepsilon(-t,\alpha)\,dt$$

whence the statement follows. □

5.2.4 Step 4: The Wiener-Hopf equation

The expression in (5.29) has the same form as in Section 4.2. We set $\gamma=i$ and $K=P_\varepsilon(\alpha)P_\varepsilon^+(\alpha)$ in Definition 2.40.

Definition 5.14. *Let $A_\varepsilon(\alpha)$ as in (5.10), $P_\varepsilon(\alpha)$ the Riesz projection of $A_\varepsilon(\alpha)$ and $P_\varepsilon^+(\alpha)$ the pseudo inverse of $P_\varepsilon(\alpha)$ with respect to $P(0)$. Furthermore let $U_\varepsilon(t,\alpha)$ be the group generated by $-iA_\varepsilon(\alpha)$. Then*

$$G_\varepsilon(t,t';\alpha):=\begin{cases} iU_\varepsilon(t,\alpha)P_\varepsilon(\alpha)P_\varepsilon^+(\alpha)U_\varepsilon(-t',\alpha), & t\le t',\\ iU_\varepsilon(t,\alpha)[P_\varepsilon(\alpha)P_\varepsilon^+(\alpha)-\mathbb{1}]U_\varepsilon(-t',\alpha), & t>t',\end{cases}$$

is called time-ordered Green operator.[2]

We note some properties of G_ε.

Lemma 5.15. *The map $t\mapsto G_\varepsilon(t,t';\alpha)$ is strongly continuous for all $t\neq t'$ and all $t'\in\mathbb{R}$. The operator $G_\varepsilon(t,t+0;\alpha)$ is trace class.*

Proof. The strong continuity follows immediately from the definition of G_ε and the strong continuity of $U_\varepsilon(t,\alpha)$. Since $P_\varepsilon^+(\alpha)=P(0)P_\varepsilon^+(\alpha)$ has finite dimensional range and is therefore trace class (Theorem 2.3) it follows that $G(t,t+0;\alpha)$ is trace class, too. □

Again we have two boundary conditions namely a usual one and an asymptotic one.

[2]This operator too is called time-ordered. This is justified because its structure is similar to that of Section 4.2.

Lemma 5.16. *The time-ordered Green operator $G_\varepsilon(t,t';\alpha)$ according to Definition 5.14 satisfies the boundary condition:*

$$P(0)G_\varepsilon(0,t';\alpha) = 0, \ -\infty < t' \le 0, \ \alpha \in [0,1], \tag{5.31}$$

and for $\varepsilon > 0$ the estimate

$$\|G_\varepsilon(t,t';\alpha)\| \le C(\varepsilon,t',\alpha)e^{\delta t}, \ t \le t' \le 0, \tag{5.32}$$

with $\delta = \delta(\varepsilon) > 0$ and $C(\varepsilon,t') \ge 0$. In particular the integral

$$\int_0^1 \int_{-\infty}^0 e^{\eta t} \operatorname{tr} BG_\varepsilon(t,t+0;\alpha)\, dt\, d\alpha$$

exists.

Proof. We note $P(0)P_\varepsilon(\alpha)P_\varepsilon^+(\alpha) = P(0)$:

$$\begin{aligned} P(0)G_\varepsilon(0,t';\alpha) &= iP(0)U_\varepsilon(0)[P_\varepsilon(\alpha)P_\varepsilon^+(\alpha) - \mathbb{1}]U_\varepsilon(-t',\alpha) \\ &= i[P(0)P_\varepsilon(\alpha)P_\varepsilon^+(\alpha) - P(0)]U_\varepsilon(-t',\alpha) \\ &= i[P(0) - P(0)]U_\varepsilon(-t',\alpha) \\ &= 0. \end{aligned}$$

For $t \le t'$ we have $G_\varepsilon(t,t';\alpha) = iU_\varepsilon(t,\alpha)P_\varepsilon(\alpha)P_\varepsilon^+(\alpha)U_\varepsilon(-t',\alpha)$. In order to prove the estimate (5.32) it suffices to consider $U_\varepsilon(t,\alpha)P_\varepsilon(\alpha)$. The statement then follows from the behaviour of $U_\varepsilon(t,\alpha)$ at infinity in Lemma 5.6. This gives the existence of the integral since $P_\varepsilon^+(\alpha) = P(0)P_\varepsilon^+(\alpha)$ und $\dim \operatorname{ran} P(0) = N$ (Theorem 2.3). ☐

For $\alpha = 0$ we recover the free time-ordered Green operator.

Lemma 5.17. *We have $G_\varepsilon(t,t';0) = e^{-\varepsilon|t-t'|}G_0(t-t')$. Here G_0 is the free time-ordered Green operator as in Theorem 4.1.*

Proof. By Definition 5.14 we have:

$$G_\varepsilon(t,t';0) = iU_\varepsilon(t,0)P_\varepsilon(0)P_\varepsilon^+(0)U_\varepsilon(-t',0), \ t \le t'.$$

By Lemma 2.17 we have $P_\varepsilon(0) = P(0)$. We compute the group $U_\varepsilon(t,0)$ with the aid of Theorem 2.37:

$$U_\varepsilon(t,0) = U_0(t)\tilde{U}_\varepsilon(t) = \tilde{U}_\varepsilon(t)U_0(t)$$

$\tilde{U}_\varepsilon(t)$ being the group generated by εJ. Thus we find:

$$G_\varepsilon(t,t';0) = i\tilde{U}_\varepsilon(t-t')P(0)U_0(t-t'), \ t \le t'.$$

Since $J^2 = \mathbb{1}$ we obtain by Lemma 2.39:

$$\tilde{U}_\varepsilon(t-t')P(0) = e^{\varepsilon(t-t')}P(0).$$

Thus we have:

$$G_\varepsilon(t,t';0) = e^{\varepsilon(t-t')}G_0(t-t'), \ t \le t'.$$

For $t > t'$ we obtain analogously

$$G_\varepsilon(t,t';0) = e^{\varepsilon(t'-t)}G_0(t-t').$$

The theorem is proven. ☐

We check the Wiener-Hopf equation for G_ε.

Theorem 5.18. *For sufficiently small $\varepsilon > 0$ the time-ordered Green operator G_ε satisfies the Wiener-Hopf equation:*

$$G_\varepsilon(t,t';\alpha)\varphi = e^{-\varepsilon|t-t'|}G_0(t-t')\varphi + \alpha\int_{-\infty}^{0} e^{-\varepsilon|t-\tau|}G_0(t-\tau)BG_\varepsilon(\tau,t';\alpha)\varphi\,d\tau,$$
$$-\infty < t,t' \leq 0,\ \alpha\in[0,1],\ \varphi\in\mathcal{H}. \quad (5.33)$$

Here G_0 is the free time-ordered Green operator as in Theorem 4.1.

Proof. We employ the integral equation (2.33) from Theorem 2.42. The necessary behaviour as $t\to-\infty$ is satisfied by 5.16. We set $\gamma = i$ and $S_1 = -iB$ in (2.33) and obtain:

$$\begin{aligned} &G_\varepsilon(t,t';\alpha)\varphi - e^{-\varepsilon|t|}G_0(t)G_\varepsilon(0,t';\alpha)\varphi \\ &\qquad = e^{-\varepsilon|t-t'|}G_0(t-t')\varphi + \alpha\int_{-\infty}^{0} e^{-\varepsilon|t-\tau|}G_0(t-\tau)BG_\varepsilon(\tau,t';\alpha)\varphi\,d\tau,\ 0<t,t'\leq 0 \end{aligned}$$

for all $\varphi\in\mathcal{H}$. Up to one summand this is the integral equation we are looking for. But this summand vanishes because of the boundary condition (5.31):

$$G_0(t)G_\varepsilon(0,t';\alpha) = iU_0(t)PG_\varepsilon(0,t';\alpha) = 0,\ t'\leq 0.$$

The theorem is proven. ☐

Theorem 5.1 now follows by combining Theorems 5.13 and 5.18.

5.3 Solution theory of the Wiener-Hopf equation

We examine the Wiener-Hopf equation (5.1). In the sequel we use the notations and assumptions of Theorem 5.1.[3] We abbreviate $U_{0,\varepsilon}(t) := U_\varepsilon(t,0)$ and $G_{0,\varepsilon}(t) := e^{-\varepsilon|t|}G_0(t)$.

First of all we show that the homogeneous equation is related to a boundary value problem.

Theorem 5.19. *Let $\varphi(\cdot)$ be a continuously differentiable solution to the integral equation:*

$$\varphi(t) = \alpha\int_{-\infty}^{0} G_{0,\varepsilon}(t-\tau)B\varphi(\tau)\,d\tau \quad (5.34)$$

with $\int_{-\infty}^{0}\|\varphi(\tau)\|\,d\tau < \infty$ and $\lim\limits_{t\to-\infty}\|\varphi(t)\| = 0$. Then $\varphi(\cdot)$ also solves the boundary value problem:

$$i\frac{d}{dt}\varphi(t) = A_\varepsilon(\alpha)\varphi(t),\ P(0)\varphi(0) = 0,\ \varphi(-\infty) = 0. \quad (5.35)$$

[3]In this section the parameter α could take arbitrary complex values. But this is not necessary for our purposes.

Proof. We split up the integral in (5.34):

$$\begin{aligned} I(t) &:= \int_{-\infty}^{0} G_{0,\varepsilon}(t-\tau)B\varphi(\tau)\,d\tau \\ &= \int_{-\infty}^{t} iU_{0,\varepsilon}(t-\tau)[P(0)-\mathbb{1}]B\varphi(\tau)\,d\tau + \int_{t}^{0} iU_{0,\varepsilon}(t-\tau)P(0)B\varphi(\tau)\,d\tau. \end{aligned}$$

We assumed $\varphi(\cdot)$ to be continuously differentiable. Then the functions $\tau \mapsto [P(0)-\mathbb{1}]B\varphi(\tau)$ and $\tau \mapsto P(0)B\varphi(\tau)$ are continuously differentiable, too, because $P(0)$ and B are bounded. It follows from Theorem 2.26 that $I(\cdot)$ is continuously differentiable and that $\varphi(\cdot)$ satisfies the differential equation (5.35). Since for all $-\infty < \tau \leq 0$ by definition we have $P(0)G_0(-\tau) = iP(0)[P(0)-\mathbb{1}]U_0(-\tau) = 0$ the boundary condition follows. □

The requirement that $\varphi(\cdot)$ is continuously differentiable cannot be dropped in general (see Section 4.3 and [11], Section 4.2). But if all operators are bounded the assumption can be weakened.

Corollary 5.20. *In addition to the assumptions of Theorem 5.1 let $A_{0,\varepsilon}$ be bounded. Let $\varphi(\cdot)$ be a continuous, integrable solution to the integral equation (5.34) on $]-\infty,0]$ Then $\varphi(\cdot)$ is differentiable and satisfies the boundary value problem (5.35).*

Proof. The group $U_{0,\varepsilon}(t)$ is generated by the bounded operator $-iA_{0,\varepsilon}$ and is therefore differentiable in the operator norm (Theorem 2.27). Then the statement follows from the proof of Theorem 5.19. □

Theorem 5.19 can be conversed.

Theorem 5.21. *Let the function $\varphi(\cdot) : t \mapsto \varphi(t) \in \mathcal{H}$ be continuously differentiable. Moreover let $\varphi(0) \in D(A_\varepsilon(\alpha))$ and $\int_{-\infty}^{0} \|\varphi(\tau)\|\,d\tau < \infty$. If $\varphi(\cdot)$ is a solution to the boundary value problem (5.35) then $\varphi(\cdot)$ satisfies the integral equation (5.34).*

Proof. Let $\varphi(\cdot)$ be a solution to the boundary value problem (5.35). By Theorem 2.25 $\varphi(t)$ can be written as

$$\varphi(t) = U_\varepsilon(t,\alpha)\varphi(0).$$

This yields together with Theorem 2.32:

$$\varphi(t) = U_{0,\varepsilon}(t)\varphi(0) + i\alpha \int_{t}^{0} U_{0,\varepsilon}(t-\tau)B\varphi(\tau)\,d\tau \tag{5.36}$$

$$\varphi(t) = -i\alpha \int_{-\infty}^{t} U_{0,\varepsilon}(t-\tau)B\varphi(\tau)\,d\tau. \tag{5.37}$$

We multiply (5.36) by $P(0)$ and (5.37) by $\mathbb{1} - P(0)$ and note the boundary condition (5.35). Then we get:

$$P(0)\varphi(t) = 0 + i\alpha \int_{t}^{0} P(0)U_{0,\varepsilon}(t-\tau)B\varphi(\tau)\,d\tau \tag{5.38}$$

$$(\mathbb{1} - P(0))\varphi(t) = i\alpha \int_{-\infty}^{t} (P(0)-\mathbb{1})U_{0,\varepsilon}(t-\tau)B\varphi(\tau)\,d\tau. \tag{5.39}$$

Addition of (5.38) and (5.39) yields the statement. □

At this point the analogy with Section 4.3 ends because we cannot connect the solution theory of the boundary value problem with the determinant $\det P(0)P_\varepsilon(\alpha)P(0)$. This is due to the fact that strictly speaking our boundary value problem is none because the second boundary point is missing. If vanishing of the solution $\varphi(\cdot)$ at infinity implied that $\varphi(\cdot)$ has the form

$$\varphi(t) = U_\varepsilon(t,\alpha)\varphi(0) = P_\varepsilon(\alpha)U_\varepsilon(t,\alpha)\varphi(0)$$

then we could continue exactly as in Section 4.3. Unfortunately this is not possible without further effort if, for example, $A_\varepsilon(\alpha)$ is unbounded and the spectrum contains continuous parts. But if $A_\varepsilon(\alpha)$ has a purely discrete spectrum consisting of simple eigenvalues then one can prove for $U_\varepsilon(t,\alpha)(\mathbb{1} - P_\varepsilon(\alpha))$ similar estimates as for $U_\varepsilon(t,\alpha)P_\varepsilon(\alpha)$ in Theorem 5.10 by using expansions into eigenfunctions. The solution theory then formally corresponds to that of Section 4.3. We do not want to stress this point.

6 Quantitative results

Let $H : D(H) \to \mathcal{H}$ be a self-adjoint operator, $U(t)$ the group of unitary operators generated by $-iH$ (see Corollary 2.24), and $P : \mathcal{H} \to \mathcal{H}$ an orthogonal projection having an N-dimensional range with $1 \leq N < \infty$. Moreover, for every $\varphi \in D(H)$ let $P\varphi \in D(H)$. We decompose H into a diagonal part and an off-diagonal part with respect to P:

$$H = H^{||} + H^{\perp} \tag{6.1}$$

with the operators

$$H^{||} := PHP + (\mathbb{1} - P)H(\mathbb{1} - P), \; H^{\perp} := PH(\mathbb{1} - P) + (\mathbb{1} - P)HP. \tag{6.2}$$

In what follows we assume $H^{||}$ to be self-adjoint on $D(H)$ and $H^{\perp}$ to be bounded. We have:

$$PH^{||} = H^{||}P \text{ und } PH^{\perp}P = 0. \tag{6.3}$$

By $H(\alpha)$ we denote the operators:

$$H(\alpha) := H^{||} + \alpha H^{\perp}, \; \alpha \in [0, 1]. \tag{6.4}$$

All $H(\alpha)$ are self-adjoint on $D(H)$. Let $U(t, \alpha)$ be the group generated by $-iH(\alpha)$ (see Corollary 2.24). Let furthermore $\mathcal{H}_1 := \operatorname{ran} P$ and $\mathcal{H}_2 := \mathcal{H}_1^{\perp}$. Then $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and H can be written as block matrix:

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \; H_{jk} : \mathcal{H}_k \to \mathcal{H}_j, \; j, k = 1, 2. \tag{6.5}$$

We assume the spectra of H_{11} and H_{22} to be disjoint.

6.1 Section determinants of unitary operators

We consider $\det PU(T, \alpha)P$. The most simple property follows from Hadamard's determinant inequality.

Theorem 6.1. *Let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator. The section determinants can be estimated by:*

$$|\det PUP| \leq 1.$$

Proof. Let u_j be the column vectors of the finite dimensional operator $PUP : \mathcal{H}_1 \to \mathcal{H}_1$. Hadamard's inequality yields:

$$|\det PUP| \leq \prod_{j=1}^{N} \|u_j\|.$$

Here $\|\cdot\|$ denotes the euclidean norm in $\mathbb{C}^N$. The columns of U form an orthonormal system. Therefore we have $\|u_j\| \leq 1$. □

We want to apply the results of Chapter 4. To this end we put $B = H^{\perp}$ in Corollary 4.12 and examine whether the homogeneous integral equation

$$\varphi(t) = \alpha \int_0^T G_0(t-\tau) H^{\perp} \varphi(\tau)\, d\tau,\ t \in [0,T],\ \alpha \in [0,1], \tag{6.6}$$

only possesses the trivial solution. Here G_0 is the free time-ordered Green operator as defined in Theorem 4.1 with $A_0 = H^{||}$. The most simple solution criterion can be achieved if $H^{\perp}$ is small.

Theorem 6.2. *Let the function $\varphi(\cdot) : t \mapsto \varphi(t) \in \mathcal{H}$ be continuous and solve the homogeneous integral equation (6.6). If $T\|H^{\perp}\| < 1$ then $\varphi(\cdot) \equiv 0$.*

Proof. Let $\varphi(\cdot)$ be a non-trivial solution to (6.6). By Lemma 4.6 we have $\|G_0(t)\| \leq 1$ since $H^{||}$ is self-adjoint. Since $\alpha \in [0,1]$ it follows:

$$\|\varphi(t)\| \leq \int_0^T \|G_0(t-\tau)\| \|H^{\perp}\| \|\varphi(\tau)\|\, d\tau \leq T\|H^{\perp}\| \sup_{\tau\in[0,T]} \|\varphi(\tau)\| < \sup_{\tau\in[0,T]} \|\varphi(\tau)\|$$

yielding a contradiction. Thus (6.6) only has the trivial solution. □

We draw a simple conclusion.

Corollary 6.3. *In (6.6) let $T\|H^{\perp}\| < 1$. Then* $\det PU(T,\alpha)P \neq 0$ *for all $\alpha \in [0,1]$.*

Proof. Theorem 6.2 and Corollary 4.12. □

The example in 4.1 suggests that the gap between the diagonal entries H_{11} and H_{22} is important rather than the smallness of $H^{\perp}$. Before proving this we have to evaluate an integral involving the free Green operator.

Lemma 6.4. *The free Green operator G_0 satisfies:*

$$\begin{aligned}\int_0^T & G_0(t-\tau) H^{\perp} G_0(\tau - t')\varphi\, d\tau \\ &= iG_0(t-T) X G_0(T-t')\varphi - iG_0(t) X G_0(-t')\varphi + X G_0(t-t')\varphi - G_0(t-t') X \varphi\end{aligned}$$

for all $\varphi \in \mathcal{H}$, X being the Friedrichs integral:

$$X = \frac{1}{2\pi i} \int_\Gamma (z\mathbb{1} - H^{||})^{-1} [H^{\perp}, P] (z\mathbb{1} - H^{||})^{-1}\, dz.$$

The path $\Gamma \in \rho(H^{||})$ encloses $\sigma(H_{11})$ but no points of $\sigma(H_{22})$.

Proof. We have assumed the spectra of H_{11} and H_{22} to be disjoint. Therefore X is well-defined. By Theorem 2.19 we have $[X, H^{||}] = [[H^{\perp}, P], P] = H^{\perp}$. Thus:

$$\int_0^T G_0(t-\tau) H^{\perp} G_0(\tau - t')\varphi\, d\tau = \int_0^T G_0(t-\tau) [X, H^{||}] G_0(\tau - t')\varphi\, d\tau.$$

Let $t \leq t'$. We split up the integral:

$$\int_0^T = \int_0^t + \int_t^{t'} + \int_{t'}^T .$$

For $t \neq 0$, $G_0(t)$ is strongly differentiable and satisfies the differential equation:

$$i\frac{d}{dt}G_0(t)\varphi = H^{||}G_0(t)\varphi,\ \varphi \in \mathcal{H}$$

(Theorem 2.41). Since $X\varphi \in D(H^{||})$ (Lemma 2.19) the function $\tau \mapsto G_0(t-\tau)XG_0(\tau - t')$ is strongly differentiable for $\tau \neq t, t'$ with:

$$i\frac{d}{d\tau}G_0(t-\tau)XG_0(\tau-t')\varphi = G_0(t-\tau)[X, H^{||}]G_0(\tau-t')\varphi,\ \tau \neq t, t'.$$

Hence it follows (all equations have to be applied on $\varphi \in D(H^{||})$):

$$\begin{aligned}
&\int_0^T G_0(t-\tau)H^{\perp}G_0(\tau-t')\,d\tau \\
&= i\left[G_0(t-\tau)XG_0(\tau-t')\right]_{\tau=0}^{\tau=t} \\
&\quad + i\left[G_0(t-\tau)XG_0(\tau-t')\right]_{\tau=t}^{\tau=t'} \\
&\quad + i\left[G_0(t-\tau)XG_0(\tau-t')\right]_{\tau=t'}^{\tau=T} \\
&= i\big(G_0(+0)XG_0(t-t') - G_0(t)XG_0(-t') \\
&\quad + G_0(t-t')XG_0(-0) - G_0(-0)XG_0(t-t') \\
&\quad + G_0(t-T)XG_0(T-t') - G_0(t-t')XG_0(+0)\big) \\
&= i\big(i(P-\mathbb{1})XG_0(t-t') - G_0(t)XG_0(-t') \\
&\quad + iG_0(t-t')XP - iPXG_0(t-t') \\
&\quad + G_0(t-T)XG_0(T-t') - iG_0(t-t')X(P-\mathbb{1})\big) \\
&= i\left(-iXG_0(t-t') + iG_0(t-t')X + G_0(t-T)XG_0(T-t') - G_0(t)XG_0(-t')\right).
\end{aligned}$$

Since $D(H^{||}) = D(H)$ is dense in $\mathcal{H}$ the statement follows for $t \leq t'$. The case $t > t'$ is treated in the same way. ☐

We apply this to the homogeneous equation.

Theorem 6.5. *Let the function $\varphi(\cdot) : t \mapsto \varphi(t) \in \mathcal{H}$ be continuous and solve the homogeneous integral equation (6.6). If $4T\|X\|\|H^{\perp}\| < 1$ then $\varphi(\cdot) \equiv 0$.*

Proof. Let $\varphi(\cdot)$ be a non-trivial solution to (6.6). Iterating (6.6) yields:

$$\begin{aligned}
\varphi(t) &= \alpha^2 \int_0^T \int_0^T G_0(t-\tau_1)H^{\perp}G_0(\tau_1-\tau)H^{\perp}\varphi(\tau)\,d\tau_1\,d\tau \\
&= \alpha^2 \int_0^T (iG_0(t-T)XG_0(T-\tau) - iG_0(t-0)XG_0(t_0-\tau) \\
&\quad + XG_0(t-\tau) - G_0(t-\tau)X)\,H^{\perp}\varphi(\tau)\,d\tau.
\end{aligned}$$

Since $\|G_0(t)\| \leq 1$ (Lemma 4.6) and $\alpha \in [0,1]$ estimating yields:

$$\|\varphi(t)\| \leq 4T\|X\|\|H^\perp\| \sup_{\tau\in[0,T]} \|\varphi(\tau)\| < \sup_{\tau\in[0,T]} \|\varphi(\tau)\|.$$

This gives a contradiction. Thus (6.6) only has the trivial solution. $\square$

A simple conclusion is

Corollary 6.6. *In (6.6) let* $4T\|X\|\|H^\perp\| < 1$. *Then* $\det PU(T,\alpha)P \neq 0$ *for all* $\alpha \in [0,1]$.

Proof. Theorem 6.5 and Corollary 4.12. $\square$

We give concrete operators whose spectrum has the necessary properties in order to define the Friedrichs integral in Theorem 6.5.

Theorem 6.7. *Let* $H_0 : D(H_0) \to \mathcal{H}$ *be self-adjoint and possess a complete orthonormal system of eigenvectors* $(\varphi_j)_{j\in\mathbb{N}}$ *with corresponding eigenvalues:*

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots.$$

Define the orthogonal projections P_N *by:*

$$P_N = \sum_{j=1}^{N} (\varphi_j, \cdot)\varphi_j.$$

Let $V : \mathcal{H} \to \mathcal{H}$ *be symmetric and bounded with* $2\|V\| < \lambda_{N+1} - \lambda_N$ *for a certain* N. *Finally put* $H := H_0 + V$. *Then the spectra of* $H_1 := P_N H P_N : \operatorname{ran} P_N \to \operatorname{ran} P_N$ *and* $H_2 := (\mathbb{1} - P_N)H(\mathbb{1} - P_N) : \operatorname{ran}(\mathbb{1} - P_N) \to \operatorname{ran}(\mathbb{1} - P_N)$ *are disjoint.*

Proof. Since P_N are spectral projections (and thus Riesz projections) of H it suffices by Theorem 2.13 to consider H_1 and H_2 on the spaces $\mathcal{H}_1 := \operatorname{span}\{\varphi_1, \dots, \varphi_N\}$ and $\mathcal{H}_2 := \overline{\operatorname{span}}\{\varphi_{N+1}, \dots\}$, respectively. We consider the quadratic form:

$$(\varphi, H_1\varphi) = (\varphi, PH_0P\varphi) + (\varphi, PVP\varphi) = \sum_{j=1}^{N} \lambda_j |(\varphi_j, \varphi)|^2 + (\varphi, PVP\varphi) \leq \lambda_N \|\varphi\|^2 + \|V\|\|\varphi\|^2$$

for $\varphi \in \mathcal{H}_1$. The self-adjointness of H_1 implies:

$$\sup_{\mu\in\sigma(H_1)} \mu \leq \lambda_N + \|V\|.$$

Analogous calculations show for H_2:

$$\inf_{\mu\in\sigma(H_2)} \mu \geq \lambda_{N+1} - \|V\|.$$

Together with the assumption on $\|V\|$ this gives the statement. $\square$

6.2 Perturbation expansion of Riesz projections

We consider the Riesz projections of $H(\alpha)$ with the aid of perturbation theory. In our presentation we shall concentrate onto the purely formal character of the calculations and neglect convergence and related questions.[1] We define the Riesz projections as usual:

$$P(\alpha) = \frac{1}{2\pi i}\int_\Gamma (z\mathbb{1} - H(\alpha))^{-1}\, dz. \tag{6.7}$$

Here the integration path Γ encloses the spectrum of H_{11} but not the spectrum of H_{22} what is possible since we assumed the two spectra to be disjoint. We assume that for all $\alpha \in [0,1]$, $\Gamma \subset \rho(H(\alpha))$[2] We have $P(0) = P$. We want to study

$$P_{11}(\alpha) := PP(\alpha)P : \mathcal{H}_1 \to \mathcal{H}_1. \tag{6.8}$$

To this end we start from the resolvent equation:

$$(z\mathbb{1} - H(\alpha))^{-1} = (z\mathbb{1} - H^{||})^{-1} + \alpha(z\mathbb{1} - H^{||})^{-1}H^{\perp}(z\mathbb{1} - H(\alpha))^{-1}. \tag{6.9}$$

Iterating this gives the perturbation expansion

$$(z\mathbb{1} - H(\alpha))^{-1} = (z\mathbb{1} - H^{||})^{-1} + (z\mathbb{1} - H^{||})^{-1}\sum_{j=1}^{\infty}\alpha^j\left[H^{\perp}(z\mathbb{1} - H^{||})^{-1}\right]^j. \tag{6.10}$$

This series is the starting point for of perturbation theory and has been examined many times (see for example [14] or [10]). We multiply (6.7) from both sides by P and insert the series (6.10). Interchanging summation and integration yields:

$$PP(\alpha)P = \sum_{j=0}^{\infty}\frac{\alpha^j}{2\pi i}\int_\Gamma P(z\mathbb{1} - H^{||})^{-1}\left[H^{\perp}(z\mathbb{1} - H^{||})^{-1}\right]^j P\, dz. \tag{6.11}$$

(6.3) implies that in (6.11) only each second summand does not vanish. Therefore it is more convenient in what follows to work with the operators H_{jk} from (6.5):

$$P_{11}(\alpha) = \mathbb{1} + \sum_{j=1}^{\infty}\alpha^{2j}I_j. \tag{6.12}$$

Here

$$I_j := \frac{1}{2\pi i}\int_\Gamma (z\mathbb{1} - H_{11})^{-1}Z(z\mathbb{1} - H_{11})^{-1}\, dz,\ j \in \mathbb{N}, \tag{6.13}$$

with

$$Z := H_{12}(z\mathbb{1} - H_{22})^{-1}H_{21}(z\mathbb{1} - H_{11})^{-1}H_{12}\cdots H_{12}(z\mathbb{1} - H_{22})^{-1}H_{21}.$$

$(z\mathbb{1} - H_{22})^{-1}$ appears in Z exactly j times. We want to estimate I_j. The operator $(z\mathbb{1} - H_{11})^{-1} : \mathcal{H}_1 \to \mathcal{H}_1$ is finite dimensional, normal, and thus diagonizable. Let $\varkappa_1, \ldots, \varkappa_N$

[1] This is justified the more because recent investigations of the author recommend methods not based on perturbation theory. See the corresponding footnote 5 in the introduction.

[2] A sufficient condition under which this is satisfied may be derived from Theorem 6.7.

be the eigenvalues of H_{11} and $E_1, \dots, E_N$ the corresponding spectral projections. Then we have:

$$(z\mathbb{1} - H_{11})^{-1} = \sum_{k=1}^{N} \frac{1}{z - \varkappa_k} E_k. \tag{6.14}$$

We insert this decomposition into (6.13):

$$\begin{aligned} I_j = \sum_{k_1=1}^{N} \cdots \sum_{k_{j+1}=1}^{N} \frac{1}{2\pi i} \int_\Gamma & \frac{1}{(z - \varkappa_{k_1}) \cdots (z - \varkappa_{k_{j+1}})} \\ & \times E_{k_1} H_{12} (z\mathbb{1} - H_{22})^{-1} H_{21} E_{k_2} H_{12} \cdots H_{12} (z\mathbb{1} - H_{22})^{-1} H_{21} E_{k_{j+1}} \, dz. \end{aligned} \tag{6.15}$$

We consider the following general integrals:

$$I_{m,n} := \frac{1}{2\pi i} \int_\Gamma \frac{1}{(z - \lambda_1) \cdots (z - \lambda_m)} A_1 R(z) A_2 \cdots A_n R(z) \, dz, \;\; m, n \in \mathbb{N}, \tag{6.16}$$

$$I_{m,0} := \frac{1}{2\pi i} \int_\Gamma \frac{1}{(z - \lambda_1) \cdots (z - \lambda_m)} \mathbb{1}, \;\; m \in \mathbb{N}. \tag{6.17}$$

Here $R(z) := (z\mathbb{1} - H_{22})^{-1}$ and $A_k : \mathcal{H}_2 \to \mathcal{H}_2$ are bounded operators. The identity in (6.17) is defined on $\mathcal{H}_2$. λ_k, $k \in \mathbb{N}$, are arbitrary complex numbers all lying inside of Γ. We recover I_j by putting essentially $A_1 = \mathbb{1}$ and $A_2 = H_{21} E_{k_2} H_{12}$ and so on.[3] The resolvent equation

$$R(z) - R(w) = (w - z) R(z) R(w) \tag{6.18}$$

yields a recursion formula for $I_{m,n}$.

Lemma 6.8. *For $I_{m,n}$ as defined in (6.16) and (6.17) we have:*

$$\begin{aligned} I_{m,n} &= I_{m,n-1} A_n R(\lambda_m) - I_{m-1,n} R(\lambda_m), \;\; m, n \geq 1, \\ I_{m,0} &= 0, \;\; m \geq 2, \;\; I_{1,n} = A_1 R(\lambda_1) \cdots A_n R(\lambda_1), \;\; n \geq 1. \end{aligned}$$

Proof. (6.9) implies for the integrand of $I_{m,n}$:

$$\begin{aligned} & \frac{1}{(z - \lambda_1) \cdots (z - \lambda_m)} A_1 R(z) \cdots A_n R(z) \\ &= \frac{1}{(z - \lambda_1) \cdots (z - \lambda_m)} A_1 R(z) \cdots A_n (R(\lambda_m) - (z - \lambda_m) R(z) R(\lambda_m)) \\ &= \frac{1}{(z - \lambda_1) \cdots (z - \lambda_m)} A_1 R(z) \cdots A_{n-1} R(z) A_n R(\lambda_m) \\ &\quad - \frac{1}{(z - \lambda_1) \cdots (z - \lambda_{m-1})} A_1 R(z) \cdots A_n R(z) R(\lambda_m). \end{aligned}$$

Integration yields the claimed formula.

For $n = 0$, $R(z)$ does not appear in the integrand of $I_{m,0}$. The function $\frac{1}{(z-\lambda_1)\cdots(z-\lambda_m)}$ has singularities only inside of Γ and behaves for $m \geq 2$ at least as $\frac{1}{|z|^2}$, $|z| \to \infty$. By the residue

[3] The factor A_1 has only been introduced in order to simplify the notation.

theorem we can deform Γ into a circle around the origin with radius r arbitrarily large. Then $r \to \infty$ yields $I_{m,0} = 0$ for $m \geq 2$.

Again by the residue theorem:

$$I_{1,n} = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z - \lambda_1} A_1 R(z) \cdots A_n R(z)\, dz = A_1 R(\lambda_1) \cdots A_n R(\lambda_1).$$

This completes the proof. $\square$

Since H_{22} is self-adjoint $\|R(z)\|$ can be computed:

$$\|R(z)\| = \frac{1}{\inf\limits_{\mu \in \sigma(H_{22})} |z - \mu|}. \tag{6.19}$$

We put

$$\Lambda := \inf_{\substack{\lambda_k,\ k \in \mathbb{N} \\ \mu \in \sigma(H_{22})}} |\lambda_k - \mu|. \tag{6.20}$$

By assumption $\Lambda > 0$. Thus from (6.19) we derive the estimate:

$$\|R(\lambda_k)\| \leq \frac{1}{\Lambda},\ k \in \mathbb{N}, \tag{6.21}$$

which can be applied to $I_{m,n}$ with the aid of the recursion formula from Lemma 6.8.

Lemma 6.9. *Let Λ be as in (6.20). We put $a_{n,j} := \|A_{j+1}\| \cdots \|A_n\|$ for $j < n$ and $a_{n,j} := 1$ for $n \leq j$. Then for all $m \geq 2$, $n \geq 1$:*

$$\|I_{m,n}\| \leq \frac{1}{\Lambda} \sum_{j=1}^{n} \frac{a_{n,j}}{\Lambda^{n-j}} \|I_{m-1,j}\|.$$

Proof. We use induction on n. $n = 1$ can be read off directly from the recursion formula:

$$I_{m,1} = I_{m,0} A_1 R(\lambda_m) - I_{m-1,1} R(\lambda_m) = -I_{m-1,1} R(\lambda_m).$$

Thus together with (6.21):

$$\|I_{m,1}\| \leq \frac{1}{\Lambda} \|I_{m-1,1}\| = \frac{a_{1,1}}{\Lambda} \|I_{m-1,1}\|.$$

The induction step goes from n to $n+1$:

$$\begin{aligned}
\|I_{m,n+1}\| &\leq \frac{\|A_{n+1}\|}{\Lambda} \|I_{m,n}\| + \frac{1}{\Lambda} \|I_{m-1,n+1}\| \\
&\leq \frac{\|A_{n+1}\|}{\Lambda} \frac{1}{\Lambda} \sum_{j=1}^{n} \frac{a_{n,j}}{\Lambda^{n-j}} \|I_{m-1,j}\| + \frac{1}{\Lambda} \|I_{m-1,n+1}\| \\
&\leq \frac{1}{\Lambda} \left[\sum_{j=1}^{n} \frac{a_{n+1,j}}{\Lambda^{n+1-j}} \|I_{m-1,j}\| + \|I_{m-1,n+1}\| \right] \\
&= \frac{1}{\Lambda} \sum_{j=1}^{n+1} \frac{a_{n+1,j}}{\Lambda^{n+1-j}} \|I_{m-1,j}\|.
\end{aligned}$$

This proves the lemma. $\square$

We simplify the last estimate.

Lemma 6.10. *Let $S_{m,n}$ defined as*

$$S_{1,n} = 1,\ n \geq 1,\ S_{m,n} = \sum_{j=1}^{n} S_{m-1,j},\ m \geq 2,\ n \geq 1.$$

Then for $m, n \geq 1$:

$$\|I_{m,n}\| \leq \frac{a_{n,0}}{\Lambda^{m+n-1}} S_{m,n}.$$

Proof. We use induction on m. For $m = 1$ we have by Lemma 6.8 and (6.21):

$$\|I_{1,n}\| \leq \frac{a_{n,0}}{\Lambda^n} = \frac{a_{n,0}}{\Lambda^n} S_{1,n}.$$

We conclude from m to $m+1$ where we start from Lemma 6.9 and note $a_{n,j}a_{j,0} = a_{n,0}$:

$$\begin{aligned}
\|I_{m+1,n}\| &\leq \frac{1}{\Lambda} \sum_{j=1}^{n} \frac{a_{n,j}}{\Lambda^{n-j}} \|I_{m,j}\| \\
&\leq \frac{1}{\Lambda} \sum_{j=1}^{n} \frac{a_{n,j}}{\Lambda^{n-j}} \frac{1}{\Lambda^{m-1}} \frac{a_{j,0}}{\Lambda^j} S_{m,j} \\
&\leq \frac{1}{\Lambda^m} \frac{a_{n,0}}{\Lambda^n} \sum_{j=1}^{n} S_{m,j} \\
&= \frac{a_{n,0}}{\Lambda^{m+n}} S_{m+1,n}.
\end{aligned}$$

This proves the lemma. □

Finally we estimate $S_{m,n}$.

Lemma 6.11. *For $m, n \geq 1$ we have:*

$$S_{m,n} \leq \frac{1}{(m-1)!}(m+n)^{m-1}.$$

Proof. We use induction on m. The case $m = 1$ is obvious:

$$S_{1,n} = 1 = \frac{1}{0!}(1+n)^0.$$

We conclude from m to $m+1$:

$$\begin{aligned}
S_{m+1,n} &= \sum_{j=1}^{n} S_{m,j} \\
&\leq \frac{1}{(m-1)!} \sum_{j=1}^{n} (m+j)^{m-1} \\
&\leq \frac{1}{(m-1)!} \int_1^{n+1} (x+m)^{m-1}\, dx \\
&= \frac{1}{m!} \left[(x+m)^m\right]_1^{n+1} \\
&\leq \frac{1}{m!}(m+1+n)^m.
\end{aligned}$$

This shows the lemma. □

We sum up the preceding lemmas.

Theorem 6.12. *Let $I_{m,n}$, $m, n \geq 1$, be as in (6.16) and Λ as in (6.20). If $a_n := \|A_1\| \cdots \|A_n\|$ then:*

$$\|I_{m,n}\| \leq \frac{a_n}{\Lambda^{m+n-1}} \frac{1}{(m-1)!} (m+n)^{m-1}.$$

Proof. Follows immediately by combining Lemmas 6.10 and 6.11 with $a_n = a_{n,0}$. □

We can use this theorem to estimate the integrals I_j we are originally interested in. We recall $N = \dim \mathcal{H}_1$.

Corollary 6.13. *Let I_j be as in (6.13) and Λ as in (6.20). Then for all $j \in \mathbb{N}$ we have:*

$$\|I_j\| \leq N^{j+1} \left(\frac{\|H_{12}\| \|H_{21}\|}{\Lambda^2} \right)^j \frac{1}{j!} (2j+1)^j.$$

Proof. The multiple sum in (6.15) consists of N^{j+1} summands each of them having the form:

$$E_{k_1} H_{12} I_{j+1,j} H_{21} E_{k_{j+1}}, \tag{6.22}$$

where λ_l in $I_{j+1,j}$ is to be chosen as

$$\lambda_l = \varkappa_{k_l}, \ l = 1, \ldots, j+1.$$

Furthermore we have $A_l = H_{21} E_{k_l} H_{12}$ for $l = 2, \ldots$ and $A_1 = \mathbb{1}$. Since E_k are orthogonal projections and hence $\|E_k\| \leq 1$, we can estimate in Theorem 6.12:

$$a_j = \|A_1\| \cdots \|A_j\| \leq (\|H_{12}\| \|H_{21}\|)^{j-1}.$$

If we note the additional factors $E_{k_1} H_{12}$ and $H_{21} E_{k_{j+1}}$ in (6.22) then the statement follows from Theorem 6.12. □

If we estimate in Corollary 6.13 the factorial with the aid of Stirling's formula then we obtain:

$$\|I_j\| \leq C q^j$$

with appropriate $C, q \geq 0$. For C, q (and α) small enough (6.12) implies that $P_{11}(\alpha)$ is invertible. This gives a connection with Chapter 5.

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