Warming up

◮ saying hi
◮ webpage: http://www.ruhr-uni-bochum.de/philosophy/defeasible-reasoning
◮ formalities

Useful Introductory Literature


Deductive Reasoning vs. Defeasible Reasoning?

Deductive Reasoning
\[
\begin{align*}
  n &> 2 \\
  n &\text{ is prime} \\
  n &\text{ is odd}
\end{align*}
\]
◮ truth-conductive:
  ◦ if each premise is true
  ◦ then the conclusion is true
  ◦ (no exceptions)

Defeasible Reasoning

◮ Tweety is a bird.
◮ Birds fly.
◮ Tweety flies.
◮ What if Tweety is a penguin?
◮ tentative
◮ not truth-conductive
◮ internal / external dynamics

What makes defeasible inferences feasible?

◮ ... and that despite the lack of truth conductiveness
◮ what “compensates” for that?
◮ nevertheless: they are “usually”, “in most cases”, “typically” or “normally” truth-conductive, e.g.
  ◦ reasoning on the basis of normality: Tweety flies since “normally” birds fly
  ◦ inductive generalizations:
    a restricted number of samples of a class of objects shares a property \( P \)
    all entities in the class share the property \( P \)
    Tacit assumption: the sample class is normal in the sense that the homogeneity of the observed property does apply to the whole class.
  ◦ probabilistic reasoning: statistical syllogism (Pollock)
    \[
    X \text{ is an } A \\
    P(A \text{ is a } B) \text{ is high} \\
    X \text{ is a } B.
    \]
    Tacit assumption: \( X \) is not exceptional with respect to the given probabilities.
The tacit normality assumption of defeasible reasoning

Premises

support

ceteris normalibus

Conclusion

The static character of non-defeasible reasoning

- Immunity to revision with respect to external information: Monotonicity
  - In terms of $\vdash$:
  - We never throw away previous inferences in face of new knowledge.
  - In terms of $Cn$:

- Immunity to revision with respect to new insights won in the reasoning process

Two types of dynamics of defeasible reasoning

- External dynamics
  - New info causes the retraction of previous inferences
  - e.g. Tweety is a penguin. $\rightarrow$ Tweety flies.
  - Pollock: synchronic defeasibility

- Internal dynamics
  - Growing insight in the given information can cause the withdrawal of previous inferences
  - Pollock: diachronic defeasibility

Premises

withdrawal

knowledge :

abnormal case

{ • due to internal dynamics
• due to external dynamics

Conclusion

Formalizing Defeasible Reasoning: Why bother?

Understanding

Unification (via Adaptive Logics)

Comparability

Finetuning

Variation

Two Types of Defeaters

Premises

undercut

rebuttal

Conclusion

- undercut: premises do not warrant conclusion
- rebuttal: conclusion does not hold

Towards ALs: a simple example

The logic $\mathbf{CL}$:
Take classical logic and add a 'dummy operator' $\circ$.
More on $\mathbf{CL}$ in a moment...
The Sherlock Holmes Twist
Interpret $\diamond A$ by “By the given evidence it is reasonable to assume $A$”.
- If our detective has reason to assume $A$, $\neg \diamond A$
- infer that $A$ is the case – defeasibly.

Is $CL_0$ already a good logic for Sherlock Holmes?
- Suppose he gets some evidence that suggests that $A$ is the case, $\neg \diamond A$.
- He cannot infer $A$ yet with $CL_0$.
- Option 1: do nothing. This would be a boring detective.
- Option 2: ‘jump to the conclusion $A$’
- however, what now if he gets different evidence that indicates that $\neg A$ is the case, $\diamond \neg A$?

How would Holmes reason?
$\diamond A$ 
$\vdash A$

defeasible assumption: $\diamond A \supset A$

How to model this formally? $\Rightarrow$ Adaptive Logics

The Three Parts that Characterize Adaptive Logics
1. The Lower Limit Logic
2. The set of abnormalities
3. The adaptive strategy

The Lower Limit Logic: in our example
interprets the given information as “normal as possible”
interprets the given information rigorously as normal

lower limit logic (LLL) 
adaptive logic (AL) 
upper limit logic (ULL)

strengthens with normality assumptions 
approximates

lower limit logic (LLL): $CL_0$
adaptive logic (AL) 
upper limit logic (ULL)

strengthens LLL with normality assumptions:
given $\diamond A$ ... assume $\diamond A \supset A$ unless ... 

$\vdash$ $\neg ULL \diamond A \supset A$
Requirements for Lower Limit Logics

- reflexive:
- transitive:
- monotonic:
- compact:
- has a characteristic semantics
- often we need to speak about an enriched LLL: it is enriched by classical operators denoted by a “check”: e.g. \(\check{\neg}\), \(\check{\lor}\) etc. (we will discuss this topic in more detail later)
- some papers make the distinction between the enriched LLL and LLL explicit by writing LLL+ for the former system.
- premise sets are considered to not contain “checked connectives”

Inference Rules and Proofs: Hilbert Style

CL is defined by Modus Ponens (MP) and the following axiom schemata:

1. \((A \supset 1)\) \(A \supset (B \supset A)\)
2. \((A \supset 2)\) \((A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))\)
3. \((A \supset 3)\) \(((A \supset B) \supset A) \supset A\)
4. \((A \land 1)\)
5. \((A \land 2)\)
6. \((A \land 3)\)
7. \((A \lor 1)\)
8. \((A \lor 2)\)
9. \((A \lor 3)\)
10. \((A \equiv 1)\) \((A \equiv B) \supset (A \equiv B)\)
11. \((A \equiv 2)\) \((A \equiv B) \supset (B \equiv A)\)
12. \((A \equiv 3)\) \(((A \equiv B) \supset (A \equiv B))\)
13. \((A \neg 1)\) \((A \equiv \neg A) \supset \neg A\)
14. \((A \neg 2)\) \((A \equiv \neg A) \supset \neg A\)
15. \((A \neg 3)\) \((A \equiv \neg A) \supset \neg A\)

Weakening
Re. by cases
Ex Contradictione Quodlibet
Excl. Middle

Example: A proof
Show: \(\{p \land q, p \supset r, r \supset s\} \vdash s\).

Task: Proof \(A \supset A\)
Tip: you only need \((A \supset 1)\) and \((A \supset 2)\).

Proof: \((B \lor C) \supset (\neg B \supset C)\)

Proof: \(A \supset B, B \supset C \vdash A \supset C\)
Idea: every proof of $M$ model:
or via an evaluation function
We concatenate
Semantic consequence: $\Gamma$
By the induction hypothesis, we have $\Gamma$
$M$
Hence, there are proofs
We show by induction on the length of
Truth-functional operator $\pi$
Let $v$
Truth in a model defined recursively:
Suppose
The Deduction Theorem
$\Gamma \cup \{A\} \vdash B$ implies $\Gamma \vdash A \supset B$.
Proof:

Resolution Theorem
The Resolution Theorem
$\Gamma \vdash A \supset B$ implies $\Gamma \cup \{A\} \vdash B$.
Proof:

The Deduction Theorem
$\Gamma \cup \{A\} \vdash B$ implies $\Gamma \vdash A \supset B$.
Proof (continued):

Resolution Theorem
The Resolution Theorem
$\Gamma \vdash A \supset B$ implies $\Gamma \cup \{A\} \vdash B$.
Proof:

Semantics: Providing Meaning
◮ atomic level: assignment function $v: A \rightarrow \{0, 1\}$
◮ model: $M$ is associated with an assignment function
◮ truth in a model defined recursively:
  - $M \models A$ where $A \in A$ iff $v(A) = 1$
  - $M \models A \land B$ iff $M \models A$ and $M \models B$
  - $M \models A \supset B$ iff $M \models A$ or $M \models B$
  - $M \models \neg A$ iff $M \not\models A$
  - $M \models A \supset B$ iff $M \models A$ or $M \not\models B$
  - $M \models A \equiv B$ iff $(M \models A$ if $M \models B)$
  - or via an evaluation function $v_M: W \rightarrow \{0, 1\}$
    - $v_M(A) = v(A)$ where $A \in A$
    - $v_M(A \land B) = \min(v_M(A), v_M(B))$
    - $v_M(A \supset B) = \max(1 - v_M(A), v_M(B))$
    - etc.
◮ Semantic consequence: $\Gamma \vdash A$ iff for all models $M$: if $M \models B$
  for all $B \in \Gamma$ then $M \models A$.
◮ Truth-functional operator $\pi$:
  $v_M(\pi(A_1, \ldots, A_n)) = f(v_M(A_1), \ldots, v_M(A_n))$ for some function $f$. 
  
Idea: every proof of $B$ from $\Gamma \cup \{A\}$ can be transformed into a proof of $A \supset B$ from $\Gamma$.
Let $P$ be an arbitrary proof of $B$ from $\Gamma \cup \{A\}$.
We show by induction on the length of $P$ that for every line $l$ in proof $P$ on which $C$ is proved, $A \supset C$ can be proven from $\Gamma$.

▶ “$l \Rightarrow l+1$”: (i) $C$ is either an axiom, or (ii) $C \in \Gamma$, or (iii) $C = A$, or (iv) $C$ is obtained via MP from $D$ and $D \supset C$ which were derived on lines $l'$ and $l''$ (where $l', l'' < l$). The first three cases are as in the induction base. Hence, suppose (iv).
▶ By the induction hypothesis, we have $\Gamma \vdash A \supset D$ and $\Gamma \vdash A \supset (D \supset C)$.
▶ Hence, there are proofs $P_1$ of $X = A \supset D$ from $\Gamma$ and $P_2$ of $Y = A \supset (D \supset C)$ from $\Gamma$.
▶ We concatenate $P_1$ and $P_2$ obtaining $P_1'$.
▶ By $(A \supset (D \supset C)) \supset ((A \supset D) \supset (A \supset C))$.
▶ By $Y$, $Z$ and MP, $W = (A \supset D) \supset (A \supset C)$.
▶ By $X$, $W$ and MP, $A \supset C$.

Proof: $\neg\neg A \supset A$
Proof:

Proof: $A \supset \neg\neg A$
Soundness
Γ ⊢ D implies Γ ⊨ D.

Proof.
- Take an arbitrary proof of D from Γ. Let M be an arbitrary model of Γ.
- We prove by induction over the length of the proof that for each formula E derived at a line l, M ⊨ E.
- "l = 1": E is either (i) an axiom or (ii) E ∈ Γ. (ii) is trivial.
  - Suppose (i). Suppose E = A ⊃ (B ⊃ A) (see (A ⊃ 1)). Let M be a model of Γ. M ⊨ A ⊃ (B ⊃ A) if v_M(A ⊃ (B ⊃ A)) = 1 if max(v_M(1 − v_M(A)), max(1 − v_M(B), v_M(A))) = 1. Note that the latter holds. The proof is similar for the other axioms.
- "l ⇒ l+1": Either (i) E is an axiom, or (ii) E ∈ Γ or (iii) E is derived via MP from F ⊃ E and F where F ⊃ E and F are derived at lines f ′ ≤ l and f″ ≤ l resp. Only (iii) is non-trivial.
  - By the induction hypothesis, M ⊨ F and M ⊨ F ⊃ E. Thus, v_M(F) = v_M(F ⊃ E) = max(1 − v_M(F), v_M(E)) = 1. Thus, M ⊨ E.

Completeness
Γ ⊨ A implies Γ ⊢ A.

Proof: we need some preparation for that.

Explosion in Hilbert
Show {A, ¬A} ⊢ B. Tip: use (A¬2) and MP.

Proposition 1
If Γ ⊬ A then Γ ∪ {¬A} is consistent.

Proof.
- Suppose Γ ∪ {¬A} is inconsistent.
  - Hence, Γ ∪ {¬A} ⊢ ¬A.
  - Hence, by the deduction theorem, Γ ⊢ ¬A ⊃ ¬¬A.
  - By (¬A1) and MP, Γ ⊢ ¬¬A.
  - Since ¬¬A ⊢ A (see above), Γ ⊢ A.

Proposition 2
If Γ is consistent then there is a model of Γ.

Proof.
- Let W be enumerated by A1, A2, . . .
- Let Γ0 = Γ and define
  \[ Γ_{i+1} = \begin{cases} Γ_i \cup \{A_l\} & \text{if } Γ_i \cup \{A_l\} \text{ is consistent} \\ Γ_i \cup \{¬A_l\} & \text{else.} \end{cases} \]
- Let Γ* = ∪i≥0 Γi.
- Claim: Γ* is consistent. Proof: by induction. IB: Γ0 is consistent by supposition. "i → i+1": by definition of Γi+1.
  - Hence, each Γi is consistent. Assume Γ* is inconsistent.
  - Hence, Γ* ⊢ A1 and Γ* ⊢ ¬A1. Hence (by compactness), there is a Γi such that Γi ⊢ A1 and there is a Γj such that Γj ⊢ ¬A1.
  - Take k = max(i, j). Then (by monotonicity), Γk ∪ {A1} and Γk ⊢ ¬A1. Hence, Γk is inconsistent (since A, ¬A ⊢ B).

Soundness
Γ ⊢ D implies Γ ⊨ D.

Proof.
- Take an arbitrary proof of D from Γ. Let M be an arbitrary model of Γ.
- We prove by induction over the length of the proof that for each formula E derived at a line l, M ⊨ E.
- "l = 1": E is either (i) an axiom or (ii) E ∈ Γ. (ii) is trivial.
  - Suppose (i). Suppose E = A ⊃ (B ⊃ A) (see (A ⊃ 1)). Let M be a model of Γ. M ⊨ A ⊃ (B ⊃ A) if v_M(A ⊃ (B ⊃ A)) = 1 if max(v_M(1 − v_M(A)), max(1 − v_M(B), v_M(A))) = 1. Note that the latter holds. The proof is similar for the other axioms.
- "l ⇒ l+1": Either (i) E is an axiom, or (ii) E ∈ Γ or (iii) E is derived via MP from F ⊃ E and F where F ⊃ E and F are derived at lines f ′ ≤ l and f″ ≤ l resp. Only (iii) is non-trivial.
  - By the induction hypothesis, M ⊨ F and M ⊨ F ⊃ E. Thus, v_M(F) = v_M(F ⊃ E) = max(1 − v_M(F), v_M(E)) = 1. Thus, M ⊨ E.

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  - By (¬A1) and MP, Γ ⊢ ¬¬A.
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- Let Γ* = ∪i≥0 Γi.
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  - Hence, each Γi is consistent. Assume Γ* is inconsistent.
  - Hence, Γ* ⊢ A1 and Γ* ⊢ ¬A1. Hence (by compactness), there is a Γi such that Γi ⊢ A1 and there is a Γj such that Γj ⊢ ¬A1.
  - Take k = max(i, j). Then (by monotonicity), Γk ∪ {A1} and Γk ⊢ ¬A1. Hence, Γk is inconsistent (since A, ¬A ⊢ B).
Define $M$ via the assignment

We show by induction over the length of a formula that $M \models A$ iff $A \in \Gamma^*$.

1. $A \in A$: by definition.
2. Let $B, C$ be such that $M \models B \mid C$ iff $B \mid C \in \Gamma^*$. Hence, $B, C \in \Gamma^*$. By the induction hypothesis, $M \models B$ and $M \models C$. Hence, $M \models B \land C$. The other way around is analogous.
3. $A = B \land C$. Let $B \land C \in \Gamma^*$. Hence, $B \in \Gamma^*$ or $C \in \Gamma^*$. By the induction hypothesis, $M \models B$ and hence $M \models \neg B$. The other way around is analogous.

Complete: $\Gamma \parallel- A$ implies $\Gamma \models A$.

Proof.

Abnormalities

They determine the normality assumptions by means of which the AL strengthens the LLL.

In other words, they determine what is means to interpret premises “normal”. (we will come to the “as possible part” later)

characterized by a logical form $F$

the set of all abnormalities denoted by $\Omega$ . . . in our example . . .

recall: the normality assumption was that if $\circ A$ then $A$ is the case

hence, $\Omega = \{\circ A \land \neg A\}$

Adaptive proofs

1. $P_1 \ldots \ldots; \text{PREM} \ \parallel$

2. $P_n \ldots \ldots; \text{PREM} \ \parallel$

1. $A_1 \ldots \ldots; \text{RU} \ \Delta_1$

2. $A_n \ldots \ldots; \text{RC} \ \Delta_1 \cup \Delta_2 \cup \Delta_3$

Adaptive proofs: the generic rules

- PREMises are introduced on the empty condition (no normality assumption is needed for that)

  If $A \in \Gamma$:

  \[
  \begin{array}{c}
  \hline
  \vdots \quad \vdots \\
  \hline
  A \\
  \hline
  \end{array}
  \]

  (PREM)

- the Unconditional Rule:

  If $A_1, \ldots, A_n \models_{\text{LLL}} B$:

  \[
  \begin{array}{c}
  \hline
  \vdots \quad \vdots \\
  \hline
  A_n \quad \Delta_n \\
  \hline
  \end{array}
  \]

  (RU)

These two rules give us the full power of the LLL:

If $\Gamma \models_{\text{LLL}} A$, then $\Gamma \models_{\text{AL}} A$.  

3. The adaptive strategy

effects both,

1. the proof theory, and
2. the semantics

Let’s start with the proof theory.
Adaptive proofs: the conditional rule

\[ A_1 \Delta_1 \]
\[ \vdots \ldots \]
\[ A_n \Delta_n \]
\[ B \Delta_1 \cup \cdots \cup \Delta_n \cup \Theta \]  

(RC)

- Dab(\Theta) is a notational convention that denotes the disjunction of abnormalities in \( \Theta \).
- where \( \Theta \subseteq \Omega \) is a finite set of abnormalities
- as in RU, the conditions of the used lines (\( \Delta_1, \ldots, \Delta_n \)) are carried forward

The reliability strategy: marking

◮ the specifics of the marking definition of an AL depend on the adaptive strategy that is used
◮ there are two standard strategies
  1. reliability strategy
  2. minimal abnormality strategy
◮ we write AL\( r \) for an AL characterized by a triple \( \langle \text{LLL}, \Omega, \text{reliability} \rangle \)
◮ we write AL\( m \) for an AL characterized by a triple \( \langle \text{LLL}, \Omega, \text{minimal abnormality} \rangle \)
Example 1: a simple case of marking

1. \( \circ A \)  \( \text{PREM } \emptyset \)
2. \( 1;RC \) \{\( \circ A \land \lnot A \)\}
3. \( A \supset B \)  \( \text{PREM } \emptyset \)
4. \( 1;RC \) \{\( \circ A \land \lnot A \)\}
5. \( \lnot A \lor \lnot B \)  \( \text{PREM } \emptyset \)
6. \( \circ B \)  \( \text{PREM } \emptyset \)
7. \( (\circ A \land \lnot A) \lor (\circ B \land \lnot B) \)  \( 1;RC \) \{\( A \land \lnot A \)\}

\( \Sigma_6(\Gamma) = \{\{\circ A \land \lnot A\}, \{\circ B \land \lnot B\}\} \)
\( U_6(\Gamma) = \{\circ A \land \lnot A\} \)

\( \Sigma_6(\Gamma) = \{\{\circ A \land \lnot A\}, \{\circ B \land \lnot B\}\} \)
\( U_6(\Gamma) = \{\circ A \land \lnot A\} \)

Some nice properties of the consequence/derivability relation

- recall: \( U_6(\Gamma) = \bigcup \Sigma_6(\Gamma) \text{ where } \Sigma_6(\Gamma) = \{\Delta \mid \text{Dab}(\Delta) \text{ is a minimal Dab-formula at stage } s\} \)
- let \( \Sigma(\Gamma) \) be the set of all \( \Delta \) for which \( \Gamma \vdash \text{LLL} \text{Dab}(\Delta) \) and for all \( \Delta' \subseteq \Delta, \Gamma \not\vdash \text{LLL} \text{Dab}(\Delta') \).
- let \( U(\Gamma) = \{\Delta \mid \Delta \subseteq \Gamma \} \)

Theorem 1

\( \Gamma \vdash_\text{AL} A \text{ iff there is a } \Delta \subseteq \Omega \text{ for which } \Gamma \vdash_\text{LLL} A \lor \text{Dab}(\Delta) \text{ and } \Delta \cap U(\Gamma) = \emptyset. \)
according to the reliability strategy lines 7 and 9 are marked.
the rationale is: since \((\varphi \land \neg \varphi)\) is a minimal Dab-formula, one of the two abnormalities is the case or even both.
in case both are the case, neither of the lines 7 and 9 is safe.

The minimal abnormality strategy

- Recall: \(\Sigma_s(\Gamma) = \{\Delta | \text{Dab}(\Delta) \text{ is a minimal Dab-formula at stage } s\}\).
- \(\Phi_s(\Gamma)\) is the set of all minimal choice sets of \(\Sigma_s(\Gamma)\).
- A choice set \(\{\Delta_i | i \in I\}\) is a set that contains a member of each \(\Delta_i (i \in I)\).
- \(\varphi\) is a minimal choice set of iff there is no choice set \(\varphi'\) such that \(\varphi' \subset \varphi\).
- example: Let \(S = \{\{1,2\}, \{2,3\}\}\).
  - \(\{1\}\) is not a choice set of \(S\) since \(\{1\} \cap \{2,3\} = \emptyset\)
  - \(\{1,2\}\) is a choice set of \(S\)
  - \(\{1,3\}\) and \(\{2\}\) are the minimal choice sets of \(S\).
- each set \(\varphi \in \Phi_s(\Gamma)\) offers a minimally abnormal interpretation of the given premises resp. minimal Dab-formulas according to the current stage of the proof \(s\). By minimally abnormal we mean that as few abnormalities as possible are interpreted as true.

So, when is a line unmarked according to minimal abnormality?

A line \(l\) with formula \(A\) and condition \(\Delta\) is not marked at stage \(s\) if:

(i) there is a \(\varphi \in \Phi_s(\Gamma)\) such that \(\varphi \cap \Delta = \emptyset\) and
(ii) for each \(\varphi \in \Phi_s(\Gamma)\) there is a \(\Delta_\varphi\) such that \(\Delta_\varphi \cap \varphi = \emptyset\) and \(A\) is derived on the condition \(\Delta_\varphi\) at stage \(s\).

What does this mean, intuitively?

- condition (i) expresses that there is a minimally abnormal interpretation \(\varphi \in \Phi_s(\Gamma)\) in which the assumption \(\Delta\) is warranted.
- condition (ii) expresses that for each minimally abnormal interpretation \(\varphi \in \Phi_s(\Gamma)\) our \(A\) is derived on an assumption \(\Delta_\varphi\) that is warranted in \(\varphi\).

Marking for minimal abnormality

A line \(l\) with conditions \(\Delta\) and formula \(A\) is marked at stage \(s\) iff:

(i) there is no \(\varphi \in \Phi_s(\Gamma)\) such that \(\varphi \cap \Delta = \emptyset\) or
(ii) for some \(\varphi \in \Phi_s(\Gamma)\) there is no line at which \(A\) is derived on a condition \(\Theta\) for which \(\Theta \cap \varphi = \emptyset\).

What does this mean, intuitively?

- condition (i) expresses that the assumption \(\Delta\) on which \(A\) is derived is not warranted in any minimally abnormal interpretation offered by \(\Phi_s(\Gamma)\), since in each \(\varphi \in \Phi_s(\Gamma)\) there is an abnormality that is also in \(\Delta\) and since the assumption expressed by the condition \(\Delta\) is that no abnormality in \(\Delta\) is true.
- condition (ii) expresses that there is a minimally abnormal interpretation \(\varphi \in \Phi_s(\Gamma)\) such that \(A\) is not derived under any condition that is warranted in \(\varphi\).

\[\begin{array}{|c|c|c|}
\hline
1 \ & 0l & \text{PREM} \ \emptyset \\
2 \ & 0j & \text{PREM} \ \emptyset \\
3 \ & \neg(l \land j) & \text{PREM} \ \emptyset \\
4 \ & l \supset m & \text{PREM} \ \emptyset \\
5 \ & j \supset m & \text{PREM} \ \emptyset \\
6 \ & l & \text{RC} \ \{0l \land \neg l\} \\
7 \ & m & \text{RU} \ \{0l \land \neg l\} \\
8 \ & j & \text{RC} \ \{0j \land \neg j\} \\
9 \ & m & \text{RU} \ \{0j \land \neg j\} \\
10 \ & (0l \land \neg l) \lor (0j \land \neg j) & 1,2,3; \text{RU} \ \emptyset \\
\hline
\end{array}\]
Floating conclusions and the adaptive strategies

Floating conclusion
A is a floating conclusion in case it is reach be various conflicting arguments.

- reliability blocks the floating conclusion m from
  \[ \Gamma = \{ o, j \wedge (l \wedge j), l \supset m, j \supset m \} \]
- minimal abnormalities derives the floating conclusion:
  \[ \Gamma \vdash_{CLm} m \]

Some nice property

- recall: \( \Phi_2(\Gamma) \) is the set of minimal choice sets of \( \Sigma_2(\Gamma) \)
- \( \Sigma(\Gamma) \) is the set of all \( \Delta \) for which \( \Gamma \vdash_{LLL} Dab(\Delta) \) and for all \( \Delta' \subseteq \Delta \), \( \Gamma \vdash_{LLL} Dab(\Delta') \).
- Let \( \Phi(\Gamma) \) be the set of minimal choice sets of \( \Sigma(\Gamma) \)

Theorem 2
\( \Gamma \vdash_{ALm} A \) iff for every \( \varphi \in \Phi(\Gamma) \) there is a \( \Delta \subseteq \Omega \) for which \( \Delta \cap \varphi = \emptyset \) and \( \Gamma \vdash_{LLL} A \lor Dab(\Delta) \).

Final derivability revisited?

- recall: A formula \( A \) is finally derived at a finite stage \( s \) at line \( l \) iff (i) \( l \) is unmarked at stage \( s \) and (ii) for every extension of the proof in which \( l \) is marked, there is a further extension in which \( l \) is unmarked.
- say: A formula \( A \) is finally *-derived at a finite stage \( s \) at line \( l \) iff (i) \( l \) is unmarked at stage \( s \) and (ii) for every finite extension of the proof in which \( l \) is marked there is a finite further extension in which \( l \) is not marked.
- Let \( \Gamma \vdash_{AL} A \) iff there is a proof in which \( A \) is finally *-derived.

Does this work?
\( \Gamma \vdash_{AL} A \) iff \( \Gamma \vdash_{AL} A \).

Nope

E.g.
\[ \Gamma = \{ (\neg A_i \land A_i) \lor (\neg A_j \land A_j) \mid j > i > 0 \} \cup \{ B \lor A_i \mid i > 1 \} \]

Here \( \Gamma \vdash_{AL} B \) while \( \Gamma \vdash_{AL} B \).

Semantics for Adaptive Logics: The basic idea

- Take the set of LLL-models of a premise set \( \Gamma \)
- order them according to their abnormal part, i.e.
  \[ Ab(M) = \{ A \in \Omega \mid M \models A \} \]
- in flat adaptive logics in standard format this is done by means of: \( M_2 < M_3 \) iff \( Ab(M_3) \subset Ab(M_2) \)
- select models that are beyond a certain threshold

What threshold?

the threshold depends on the strategy:

Minimal Abnormality

- Idea: take the minimally abnormal models

  \[ M \in M_{ALm}(\Gamma) \] if
  \[ M \in M_{LLL}(\Gamma) \] and for all \( M' \in M_{LLL}(\Gamma) \), if
  \[ Ab(M') \subseteq Ab(M) \] then
  \[ Ab(M') = Ab(M) \].

Reliability

- Idea: take models whose abnormal part only consists of unreliable abnormalities
- we call this models “reliable”

  \[ M \in M_{AL}(\Gamma) \] if
  \[ M \in M_{LLL}(\Gamma) \] and
  \[ Ab(M) \subseteq U(\Gamma) \]
Let’s go back to our example...

- $\Gamma = \{ \top_l, \top_j, \neg (l \land j), l \supset m, j \supset m \}$
- $\Gamma \vdash_T (\top_l \land \neg \top_l) \lor (\top_j \land \neg \top_j)$
- $\Gamma \not\vdash_T l \land \neg l$
- $\Gamma \not\vdash_T o \land \neg j$
- hence, $U(\Gamma) = \{ o \land \neg l, o \land \neg j \}$
- we have for instance the following models $M_1, \ldots, M_6$ where
  - $\text{Ab}(M_1) = \{ o \land \neg j \}$
  - $\text{Ab}(M_2) = \{ oj \land \neg j \}$
  - $\text{Ab}(M_3) = \{ o \land \neg l, ok \land \neg k \}$
  - $\text{Ab}(M_4) = \{ oj \land \neg j, ao \land \neg o \}$
  - $\text{Ab}(M_5) = \{ o \land \neg l, oj \land \neg j, ok \land \neg k, ao \land \neg o \}$
- models $M_1$ and $M_2$ are minimally abnormal
- models $M_1, M_2, \text{and } M_3$ are reliable

Is the ordering of models smooth?

The danger: infinite descending chains without minima

- w.r.t. the infinite chains without minima there are no minimally abnormal models
- e.g. if there are only infinite chains without minima there are no minimally abnormal models: $\Gamma \models \text{AL} \perp$ (although there are LLL-models of $\Gamma$ and hence $\Gamma \not\models \text{LLL} \perp$)

Simple facts about choice sets

Let in the following $\Sigma = \Sigma(\Gamma)$.

Fact 4
Where $\varphi$ is a choice set of $\Sigma$ and $A \in \varphi$: If $A$ satisfies

$$\therefore A \in \Sigma : \varphi \cap \Delta = \{ A \} \quad (i)$$

then $\varphi \setminus \{ A \}$ is not a choice set of $\Sigma$.

Fact 5
Where $\varphi$ is a choice set of $\Sigma$ and $A \in \varphi$: If $A$ doesn’t satisfy $(i)$ then $\varphi \setminus \{ A \}$ is also a choice set of $\Sigma$.

Fact 6
Where $\varphi$ is a choice set of $\Sigma$: each $A \in \varphi$ satisfies $(i)$ iff $\varphi$ is a minimal choice set of $\Sigma$. 

Smoothing and reassurance

- a partial order $(X, \prec)$ is well-founded iff there are no infinitely descending chains.
- a partial order $(X, \prec)$ is smooth (resp. stoppered) iff for each $x \in X$ there is a minimal element $y \in x$ such that $y \prec x$ or $y = x$
- what we need is: If $\text{Ab}(M) = M \in \mathcal{M}_{\text{LLL}}(\Gamma)$, $\subset$ is smooth. (Note it may be smooth but not well-founded (e.g. invert the order on the natural numbers))

Theorem 3
1. For every LLL-model $M$ of $\Gamma$, $M$ is minimally abnormal or there is an LLL-model $M'$ of $\Gamma$ such that $\text{Ab}(M') \subset \text{Ab}(M)$ and $M'$ is minimally abnormal.
2. If $\Gamma \not\models \text{LLL} \perp$ then $\Gamma \not\models \text{AL} \perp$
3. If $\Gamma$ has LLL-models, then there are minimally abnormal models of $\Gamma$.
4. If $\Gamma \not\models \text{LLL} \perp$ then $\Gamma \not\models \text{AL} \perp$.

Lemma 7
Where $\varphi = \{ A_1, A_2, \ldots \}$ is a choice set of $\Sigma$, let $\hat{\varphi} = \bigcap_{i \in \mathbb{N}} \varphi_i$ where $\varphi_1 = \varphi$ and

$$\varphi_{i+1} = \begin{cases} \varphi_i \setminus \{ A_i \} & \text{if there is a } \Delta \in \Sigma \text{ s.t. } \varphi_i \cap \Delta = \{ A_i \} \\ \varphi_i & \text{else} \end{cases}$$

$\hat{\varphi}$ is a minimal choice set of $\Sigma$.

Proof.
- note that $\varphi_i$ is a choice set of $\Sigma$ for each $i \in \mathbb{N}$
- Assume for some $\Delta \in \Sigma$, $\hat{\varphi} \cap \Delta = \emptyset$. Note that since $\Delta$ is finite $\Delta \cap \varphi_i = \{ B_1, \ldots, B_n \}$ for some $n \in \mathbb{N}$. Assume there no $B_j$ s.t. for all $i \in \mathbb{N}$, $B_j \notin \varphi_i \cap \Delta$. Hence, for all $B_j$’s there is a $j$ such that $\varphi_j \cap \Delta = \emptyset$. Take $k = \max(\{ j \mid 1 \leq j \leq n \})$, then $B_j \notin \varphi_k \cap \Delta$ since $(\ast)$ $\{ B_1, \ldots, B_n \} \subseteq \varphi_i \cap \Delta \supseteq \varphi_{i+1} \cap \Delta$. This is a contradiction since $\varphi_i$ is a choice set of $\Sigma$ and $(\ast)$.
- Suppose some $A_i \in \varphi$ doesn’t satisfy $(i)$. Hence, for all $\Delta \in \Sigma$, $\varphi_i \cap \Delta \neq \{ A_i \}$. Hence, $\varphi_i \cap \Delta \neq \{ A_i \}$ for all $\Delta \in \Sigma$. But then $A_i \notin \varphi_i$—a contradiction.
- Hence, by the fact above, $\hat{\varphi}$ is a minimal choice set.
Simple facts about the relation between choice sets and the abnormal parts of models

Corollary 10
Where \( M \in M_{LLL}(\Gamma) \), \( \text{Ab}(M) \not\subseteq \varphi \) for all \( \varphi \in \Phi(\Gamma) \).

Corollary 11
For all \( \varphi \in \Phi(\Gamma) \) there is a \( M \in M_{LLL}(\Gamma) \) such that (i) \( \text{Ab}(M) = \varphi \) and (ii) \( M \in M_{ALr}(\Gamma) \).

Corollary 12
Where \( M \in M_{LLL}(\Gamma) \), \( M \in M_{ALm}(\Gamma) \) iff \( \text{Ab}(M) \subseteq \varphi \).

Corollary 13 (Strong Reassurance)
For each \( M \in M_{LLL}(\Gamma) \) there is a \( M' \in M_{ALm}(\Gamma) \) such that \( \text{Ab}(M') \subseteq \text{Ab}(M) \).

Proof.
By Lemma 9 and Lemma 7 there is a \( \varphi \in \Phi(\Gamma) \) such that \( \varphi \subseteq \text{Ab}(M) \). By Corollary 11 there is a \( M' \in M_{ALm}(\Gamma) \) for which \( \text{Ab}(M') = \varphi \).

Links between the marking and the semantic selection: Reliability

Syntax

Theorem 14
\( \Gamma \vdash_{ALr} A \) iff there is a \( \Delta \subseteq \Omega \) for which \( \Gamma \vdash_{LLL} A \lor \text{Dab}(\Delta) \) and \( \Delta \cap \text{U}(\Gamma) = \emptyset \).

Semantics
\( \Gamma \models_{ALr} A \) iff (for each \( M \in M_{LLL}(\Gamma) \), if \( \text{Ab}(M) \subseteq \text{U}(\Gamma) \), then \( M \models A \)).

Conflicts in adaptive proofs

A conflict between a defeasible inference and a “hard fact”

- “hard facts”: derived on empty condition
  - Type 1: hard facts conflict with defeasible assumptions
    - \( \rightarrow \) marking
  - Type 2: hard facts conflict with defeasible conclusions
    - \( \not\rightarrow \) marking
    - shortcut rule
    - in this case \( \Gamma \vdash_{LLL} \text{Dab}(\Delta) \)
    - line will be marked

Lemma 17
An \( AL \)-proof contains a line at which \( A \) is derived on the condition \( \Delta \) iff \( \Gamma \vdash_{LLL} A \lor \text{Dab}(\Delta) \).
A conflict between two defeasible inferences

- **Type 1**: concerning the defeasible assumption
  
  \[ \Gamma \vdash_{ULL} A \quad \Gamma \vdash_{ULL} \Delta \]
  
  - in this case \( \Gamma \vdash_{ULL} \text{Dab}(\Delta \cup \Theta) \)
  - shortcut rule:
    \[
    \frac{A \quad \Delta}{\text{Dab}(\Delta \cup \Theta)} \quad (\text{RD1})
    \]

- **Type 2**: concerning defeasible consequences
  
  \[ \Gamma \vdash_{ULL} \text{Dab}(\Delta \cup \Theta) \]
  
  - in this case \( \Gamma \vdash_{ULL} \text{Dab}(\Delta \cup \Theta) \)
  - shortcut rule:
    \[
    \frac{-A \quad \Delta}{\text{Dab}(\Delta \cup \Theta)} \quad (\text{RD2})
    \]

Some trouble with the classical connectives

- we need some classical connectives in order to express Dab-formulas (i.e. the classical disjunction)
- but what if the LLL has already a classical disjunction?
- suppose \( \lor \) is classical and part of the language of the LLL

The Upper Limit Logic

- Recall: the upper limit logic rigorously interprets the premises normal
- hence, \( \Gamma \vdash_{ULL} A \) for all \( A \in \Omega \)
- the consequence relation of the upper limit logic is then defined as follows:
  \[
  \Gamma \vdash_{ULL} A \iff \Gamma \cup \{ \neg A \mid A \in \Omega \} \vdash_{LLL} A
  \]

  - semantically \( ULL \) is characterized by all LLL-models \( M \) of \( \Gamma \) that are "normal", i.e. that have an empty abnormal part, \( \Lambda b(M) = \emptyset \).
  - these are precisely the LLL-models of \( \Gamma \cup \{ \neg A \mid A \in \Omega \} \).

How to save the day?

- classical "checked" symbols are superimposed on the language of LLL
  
  - where \( W \) is the set of wffs of the LLL, \( W^+ \) is the \( (\forall, \land, \neg, \ldots) \)-closure of \( W \)
  - premise sets are considered to be formulated in \( W \)
  - sometimes authors distinguish btw. \( LLL \) and \( LLL^+ \)
  - Dab-formulas are formulated with \( \lor \)

- Why does this solve our problem?
  
  - line 4 is not marked anymore since \( !A_1 \lor !A_2 \) is not a Dab-formula

ALs approximate ULL

**Theorem 18**
\[
C_{ULL}(\Gamma) \subseteq C_{AL}(\Gamma) \subseteq C_{ULL}(\Gamma)
\]

**Definition 19**

- A premise set \( \Gamma \) is normal iff it has one of the following equivalent properties
  1. \( \Gamma \cup \{ \neg A \mid A \in \Omega \} \) is LLL-non-trivial
  2. there are LLL-models \( M \) of \( \Gamma \) that are normal, i.e. for which \( \Lambda b(M) = \emptyset \)

**Theorem 20**

- If \( \Gamma \) is normal, then \( C_{AL}(\Gamma) = C_{ULL}(\Gamma) \).
  
  If a premise set can rigorously be interpreted as normal, then the adaptive logic does so.
Properties of the Standard Format

Theorem 21 (Soundness and Completeness)
Γ ⊢ AL A iff Γ |= AL A.

Theorem 22 (Reflexivity)
Γ ⊆ CNAL (Γ)

Theorem 23 (Hierarchy of the Consequence Relations)
CNLLL (Γ) ⊆ CNAL (Γ) ⊆ CNMAL (Γ) ⊆ CNULL (Γ)

Theorem 24 (Redundancy of LLL w.r.t. AL)
CNLLL (CNAL (Γ)) = CNAL (Γ)

Theorem 25
CNAL (CNLLL (Γ)) = CNAL (Γ)

Theorem 26 (Fixed Point)
CNAL (Γ) = CNAL (CNAL (Γ))

Properties of the Standard Format

Theorem 27 (Cautious Cut / Cumulative Transitivity)
If Γ′ ⊆ CNAL (Γ) then CNAL (Γ ∪ Γ′) ⊆ CNAL (Γ).

Theorem 28 (Cautious Monotonicity)
If Γ′ ⊆ CNAL (Γ) then CNAL (Γ) ⊆ CNAL (Γ ∪ Γ′).

Corollary 29 (Cautious Indifference)
If Γ′ ⊆ CNAL (Γ) then CNAL (Γ) = CNAL (Γ ∪ Γ′).

Theorem 30 (Non-Monotonicity/Non-Transitivity)
If CNLLL (Γ) ⊂ CNAL (Γ) then AL is non-monotonic and non-transitiv.

The “rational” properties

Theorem 31
In general AL is not rational monotonous, i.e. the following does not hold:

If A ∈ CNAL (Γ) and A ∈ CNAL (Γ ∪ {B}), then B ∈ CNAL (Γ)

Theorem 32
Rational distributivity does not hold for ALs in general, i.e. the following does not hold:

If A ∉ CNAL (Γ ∪ {B}) and A ∉ CNAL (Γ ∪ {C}), then A ∉ CNAL (Γ ∪ {B ∪ C})

Other strategies: the simple strategy

- applicable in case all minimal Dab-consequences are abnormalities
- then: U(Γ) = Φ(Γ) and hence the reliability strategy and the minimal abnormality strategy result in the same consequence set
- then: all adaptive models have the same abnormal part
- simplified marking condition
- semantic selection ala minimal abnormality or reliability (both select the same models in this case)
- Task: understand why.

Definition 33 (Marking for the Simple Strategy)
A line l with condition ∆ is marked at stage s iff some B ∈ ∆ is derived on the empty condition.

Definition 34 (Marking for the Simple Strategy 2)
A line l with condition ∆ is marked at stage s iff for some ∆′ ⊆ ∆, Dab(∆′) is derived on the empty condition.

Other strategies: normal selections

- Rescher-Manor consequence relations:
  - strong: ∩ MCS(Γ)
  - weak: ∪ MCS(Γ)
- Default reasoning
- skeptical: in all extensions of the given default theory
- credulous: in some extension of the given default theory
- Abstract argumentation
  - skeptical: in all extensions of a given argumentation framework
  - credulous: in some extension of a given argumentation framework
- Adaptive Logics
  - standard format: {A | M |= A}
  - normal selections

Some open questions for you

What about some well-known weakenings of Rational Monotonicity?
- If B ∈ CNL (Γ) and ¬(B ∧ C) ∉ CNL (Γ), then B ∈ CNL (Γ ∪ {C}). (proposed by Lou Goble)
- If B ∈ CNL (Γ) and ¬B ∉ CNL (Γ ∪ {C}), then B ∈ CNL (Γ ∪ {C}). (proposed by Giordano et al.)
Normal Selections Strategy: going “weak” resp. “credulous”

Semantics

- equivalence relation on the LLL-models: \( M \sim M' \) iff \( \Lambda (M) = \Lambda (M') \)
- partition of the minimally abnormal models:

\[
\begin{array}{cccc}
[M_1]_{\sim} & [M_2]_{\sim} & [M_3]_{\sim} & \ldots \\
\end{array}
\]

- \( \Gamma \vdash_{AL} A \) iff there is a \( M \in M_{ALm}(\Gamma) \) such that for all \( M' \in [M]_{\sim}, M' \models A \).

Normal Selections: Marking

Definition 35 (Marking for Normal Selections)
A line \( l \) with condition \( \Delta \) is marked at stage \( s \) iff \( \text{Dab}(\Delta) \) is derived on the empty condition at stage \( s \).

Take \( \Gamma = \{ !A \lor !B, X \lor !A, Y \lor !A \lor !B \} \).

1. \( !A \lor !B \) \quad \text{PREM} \quad \emptyset
2. \( X \lor !A \) \quad \text{PREM} \quad \emptyset
3. \( Y \lor !A \lor !B \) \quad \text{PREM} \quad \emptyset
4. \( X \) \quad 2; \text{RC} \quad \{ !A \}
5. \( Y \) \quad 3; \text{RC} \quad \{ !A, !B \}
6. \( !A \lor !B \) \quad 1; \text{RC} \quad \emptyset

Combining ALs

1. diachronic combinations / sequential combination / vertical combination / superposing ALs
   \[ \Gamma \rightarrow AL_1 \rightarrow AL_2 \rightarrow \ldots \rightarrow \text{consequences} \]

2. synchronic combinations / horizontal combination / HAL
   \[ \begin{array}{ccc}
   AL_1 & AL_2 & AL_3 \\
   \hline
   LLL \\
   \end{array} \]
   \[ \text{consequences} \]

Sequential Combinations

Consequence sets

- finite case:
  \[ C_{n_{\text{SAL}}} (\Gamma) = C_{n_{\text{AL}_n}} \left( \ldots C_{n_{\text{AL}_1}} \left( C_{n_{\text{AL}_1}} (\Gamma) \right) \ldots \right) \]
- infinite case:
  \[ C_{n_{\text{SAL}}} (\Gamma) = C_{n_{\text{AL}_n}} \left( \ldots (C_{n_{\text{AL}_1}} \left( C_{n_{\text{AL}_1}} (\Gamma) \right)) \ldots \right) \]

This is generalized to the infinite case as follows:

\[ C_{n_{\text{SAL}}} (\Gamma) = \lim_{i \to \infty} C_{n_{\text{SAL}}} (\Gamma) = \limsup_{i \to \infty} C_{n_{\text{SAL}}} (\Gamma) \]

Normal Selections

Note: not what is valid in some adaptive model is a consequence!

\[ \Gamma = \{ !A \lor !B, X \lor !A \} \]. Minimally abnormal models:
- models with abnormal part \{ !A \}:
  - some validate \( C \) (some arbitrary non-abnormal formula)
  - some validate \( \neg C \)
- models with abnormal part \{ !B \}: these validate \( X \).

We have \( \Gamma \models_{AL} A \) but \( \Gamma \not\models_{AL} C \).

References:
- Diderik Batens’ forthcoming book
- Frederik Van De Putte, Hierarchic Adaptive Logics [Logic Journal of the IGPL, 2011]
- Frederik Van De Putte and Christian Straßer, Extending the Standard Format of Adaptive Logics to the Prioritized Case [Logique et Analyse, To appear]
- Frederik Van De Putte and Christian Straßer, Three Formats of Prioritized Adaptive Logics: a Comparative Study [Under review,]
Sequential Combinations

Semantics
- take all $\mathcal{AL}_1$-models: $M_1$
- in case $s_2 = m$ take all minimally abnormal models (w.r.t. $\Omega_2$) from $M_1$
- in case $s_2 = r$ take all reliable models (w.r.t. $\Omega_2$) from $M_1$:
  select all models $M \in M_1$ for which $\text{Ab}(M) \subseteq \{ \text{Ab}(M') | M' \in M_2^m \}$ where $M_2^m$ is the set of all minimally abnormal models (w.r.t. $\Omega_2$) from $M_1$
- this way we get $M_2$
- repeat this procedure until you're through with all the ALs in the sequence

Problems with Sequential Combinations: Lack of Deduction Theorem
Note: lack of deduction theorem is the culprit:
This also shows that we don't have Cautious Transitivity:

Problems with Sequential Combinations: No Fixed Point
Suppose we have $s_1 = s_2 = r$ and
$\Gamma = \{ !A_1 \lor !A_2, !A_1 \lor !B, X \lor !A_2 \}$ where $!A_1, !A_2 \in \Omega_1 \setminus \Omega_2$ and $!B \in \Omega_2 \setminus \Omega_1$. Take a look at the following $\mathcal{AL}_1$-proof:

Problems with Sequential Combinations

Problems with Sequential Combinations: No Fixed Point
Let $\Gamma = \{ X \lor !A_1 \lor !A_2 \}$:
Let's now apply $\mathcal{AL}_1$ to the premise set $Cn_{\mathcal{AL}_1}(\Gamma)$:

Problems with Sequential Combinations

Problems with Sequential ALs: Lack of completeness for minimal abnormality
Let $\Gamma = \{ X \lor !A_i \lor !B_i | i \in \mathbb{N} \} \cup \{ !A_i \lor !A_j | i \neq j \}$ and $A_i \in \Omega_1 \setminus \Omega_2$ and $B_i \in \Omega_2 \setminus \Omega_1$. Take a look at the following $\mathcal{AL}_1^m$-proof from $\Gamma$:

Problems with Sequential ALs: Lack of completeness for minimal abnormality
Now let's take a look at the semantic selection. $\mathcal{M}_{\mathcal{AL}_1}(\Gamma) = \{ M \in \mathcal{M}_{\mathcal{LLL}}(\Gamma) | \text{Ab}_1(M) = \{ A_i \}, i \in \mathbb{N} \}$.
Hence, for each $M \in \mathcal{M}_{\mathcal{AL}_1}(\Gamma)$, $M \models X \lor !B_i$ for some $i \in \mathbb{N}$.
$\mathcal{M}_2 = \{ M \in \mathcal{M}_{\mathcal{AL}_1}(\Gamma) | \text{Ab}_2(M) = \{ !A_i \} \}$. Hence, for all $M \in \mathcal{M}_2$, $M \models X$.
Hence $X \not\in Cn_{\mathcal{AL}_1}(\Gamma)$ but $\Gamma \models X$.
Suppose $\Omega_1 \subseteq \Omega_2 \subseteq \ldots$. Then, $\text{SAL}$ is sound.

If one of the following holds, then $\text{SAL}$ is sound and complete, has a fixed point, is cautious transitive, etc. See Ref. Let $\Sigma(\Gamma)$ and $\Phi(\Gamma)$ be the corresponding sets of the AL $\langle \text{LLL}, \bigcup_{i \in I} \Omega_i, m \rangle$.

1. $\Sigma(\Gamma)$ is finite
2. every $\varphi \in \Phi(\Gamma)$ is finite
3. $\Phi(\Gamma)$ is finite

Let $\Delta, \Delta' \in \mathcal{P}(\bigcup_{i \in I} \Omega_i)$. We write $\Delta \sqsubseteq \Delta'$ for $\langle \Delta \cap \Omega_i \rangle_{i \in I} \sqsubseteq_{\text{lex}} \langle \Delta' \cap \Omega_i \rangle_{i}$.

Prioritized ALs

Prioritized abnormalities

$\Omega_1$: contains the ones we want to avoid mostly

$\Omega_i$: having a choice between a level $i$ abnormality and higher order level abnormality we’d choose the level $i$, but would prefer higher level abnormalities

Prioritized Format for ALs

$\langle \text{LLL} \rangle$ lower limit logic

sequence of abnormalities: $\langle \Omega_i \rangle_{i \in I}$

adaptive strategy: minimal abnormality and reliability

Prioritized ALs: Semantics

Definition 36

Ab($M$) = $\{ A \in \Omega \mid M \models A \}$ where $\Omega = \bigcup_i \Omega_i$.

$M \in M_{\text{ALm}}(\Gamma)$ iff $M \in M_{\text{LLL}}(\Gamma)$ and there is no $M' \in M_{\text{LLL}}(\Gamma)$ such that Ab($M'$) ⊎ Ab($M$).

Alternative Characterization

$M \in M_{\text{ALm}}(\Gamma)$ iff $M \in M_{\text{LLL}}(\Gamma)$ and Ab($M$) ∈ $\Phi(\Gamma)$.

Prioritized ALs: Proof theory

Definition 37

A line $l$ with formula $A$ and condition $\Delta$ is marked iff (i) no $\varphi \in \Phi^{+}_i(\Gamma)$ is such that $\varphi \cap \Delta = \emptyset$, or (ii) for a $\varphi \in \Phi^{+}_i(\Gamma)$ there is no line on which $A$ is derived on a condition $\Theta$ for which $\Theta \cap \varphi = \emptyset$.

Suppose in the following that $\Omega_i \subseteq \Omega_{i+1}$ for all $i, i+1 \in I$. Then we have two more alternative semantic characterizations:

1. sequential selections:
   
   a. $M[0]$ = set of LLL-models of $\Gamma$
   
   b. for each $i$ in $I$ do
   
   c. $M[i]$ = the set of all minimal abnormal models in $M[i-1]$ w.r.t. $\Omega_i$
   
   d. $M[i] = \bigcap_{i \in I} \mathcal{M}_{\text{ALm}}(\Gamma)$.

2. intersecting: $\mathcal{M}_{\text{ALm}}(\Gamma) = \bigcap_{i \in I} \mathcal{M}_{\text{ALm}}(\Gamma)$.
Let \( \Gamma = \{ X \lor \neg A_i \lor \neg B_i \mid i \in \mathbb{N} \} \cup \{ \neg A_i \lor \neg A_j \mid i \neq j \} \) and \( A_i \in \Omega_1 \setminus \Omega_2 \) and \( B_i \in \Omega_2 \setminus \Omega_1 \).

1. \( X \lor \neg A_1 \lor \neg B_1 \) PREM \( \emptyset \)
2. \( X \) 1; RC \( \{ \neg A_1, \neg B_1 \} \)
3. \( \neg A_1 \lor \neg A_2 \) PREM \( \emptyset \)
4. \( X \lor \neg A_2 \lor \neg B_2 \) PREM \( \emptyset \)
5. \( X \) 4; RC \( \{ \neg A_2, \neg B_2 \} \)

Note that \( \Phi \equiv (\Gamma) = \{ \Omega_1 \setminus \{ \neg A_i \} \mid i \in \mathbb{N} \} \). Since we can derive \( X \) on the condition \( \{ \neg A_1, \neg B_1 \} \) for each \( i \in \mathbb{N} \), \( \Gamma \vdash_{AL} X \).

Gamma = \( \{ \neg A \lor \neg B, \neg A \lor \neg C, X \lor \neg A, Y \lor \neg B \} \). Let \( \neg A \in \Omega_1 \), \( \neg B \in \Omega_2 \setminus \Omega_1 \) and \( \neg C \in \Omega_3 \setminus (\Omega_1 \cup \Omega_2) \).

1. \( \neg A \lor \neg B \) PREM \( \emptyset \)
2. \( \neg A \lor \neg C \) PREM \( \emptyset \)
3. \( X \lor \neg A \) PREM \( \emptyset \)
4. \( Y \lor \neg B \) PREM \( \emptyset \)
5. \( X \) 3; RC \( \{ \neg A \} \)
6. \( Y \) 4; RC \( \{ \neg B \} \)
7. \( \neg A \lor \neg B \) 1; RU \( \emptyset \)
8. \( \neg A \lor \neg C \) 2; RU \( \emptyset \)

We have \( \Phi \equiv (\Gamma) = \{ \{ \neg B, \neg C \} \} \) since \( \{ \neg B, \neg C \} \subset \{ \neg A \} \).

Prioritized ALs: Meta-Theory

- very rich: similar to standard format, e.g.
- soundness and completeness
- Strong reassurance
- reflexivity
- cautious indifference
- fixed point
- if \( \Gamma \) is normal, then \( Cn_{AL} (\Gamma) = Cn_{ULL} (\Gamma) \).