

Adaptive Logics – SS2015 – @RUB

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Warming up

- ▶ saying hi
- ▶ webpage:
`http://www.ruhr-uni-bochum.de/philosophy/defeasible-reasoning/`
- ▶ formalities

Useful Introductory Literature

- ▶ Batens, D. (2004). The need for adaptive logics in epistemology. In (Eds.), Logic, epistemology, and the unity of science (pp. 459485). Springer.
- ▶ Batens, D. (2007). A universal logic approach to adaptive logics. *Logica Universalis*, 1, 221242.
- ▶ Christian Straßer (2014). Adaptive logic and defeasible reasoning. applications in argumentation, normative reasoning and default reasoning. Springer.

Deductive Reasoning vs. Defeasible Reasoning?

Deductive Reasoning

$$\frac{n > 2 \\ n \text{ is prime}}{n \text{ is odd}}$$

- ▶ truth-conductive:
 - ▶ if each premise is true
 - ▶ then the conclusion is true
 - ▶ (no exceptions)

Defeasible Reasoning

$$\frac{\text{Tweety is a bird.} \\ \text{Birds fly.}}{\text{Tweety flies.}}$$

- ▶ What if Tweety is a penguin?
- ▶ tentative
- ▶ not truth-conductive
- ▶ internal / external dynamics



What makes defeasible inferences feasible?

- ▶ ... and that despite the lack of truth conductiveness
- ▶ what “compensates” for that?
- ▶ nevertheless: they are “usually”, “in most cases”, “typically” or “normally” truth-conductive, e.g.
 - ▶ reasoning on the basis of normality: Tweety flies since “normally” birds fly.
 - ▶ inductive generalizations:

$$\frac{\text{a restricted number of samples of a class of objects shares a property } \mathcal{P}}{\text{all entities in the class share the property } \mathcal{P}}$$

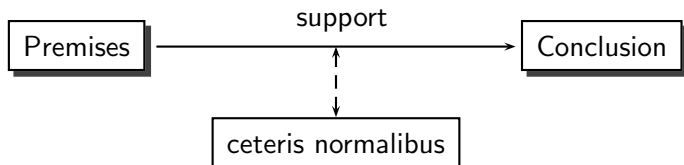
Tacit assumption: the sample class is normal in the sense that the homogeneity of the observed property does apply to the whole class.

- ▶ probabilistic reasoning: statistical syllogism (Pollock)

$$\frac{\begin{array}{l} X \text{ is an } A \\ P(A \text{ is a } B) \text{ is high} \end{array}}{X \text{ is a } B.}$$

Tacit assumption: X is not exceptional with respect to the given probabilities.

The tacit normality assumption of defeasible reasoning



The static character of non-defeasible reasoning

- ▶ Immunity to revision with respect to external information:
Monotonicity
 - ▶ In terms of \vdash :
 - ▶ We never throw away previous inferences in face of new knowledge.
 - ▶ In terms of Cn :

- ▶ Immunity to revision with respect to new insights won in the reasoning process

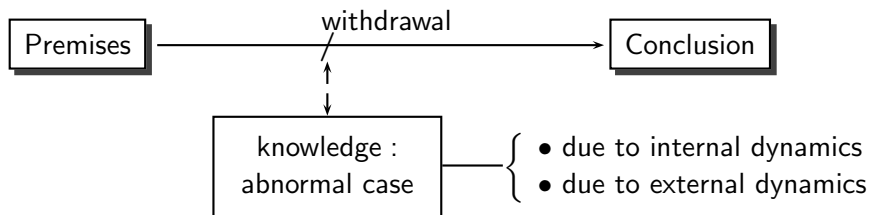
Two types of dynamics of defeasible reasoning

▶ External dynamics

- ▶ new info causes the retraction of previous inferences
- ▶ e.g. Tweety is a penguin. \rightarrow ~~Tweety flies.~~
- ▶ Pollock: *synchronic defeasibility*

▶ Internal dynamics

- ▶ growing insight in the given information can cause the withdrawal of previous inferences
- ▶ Pollock: *diachronic defeasibility*



Formalizing Defeasible Reasoning: Why bother?

Understanding

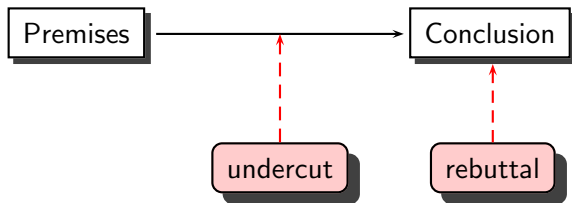
Unification (via Adaptive Logics)

Comparability

Finetuning

Variation

Two Types of Defeaters



- ▶ undercut: premises do not warrant conclusion
- ▶ rebuttal: conclusion does not hold

Towards ALs: a simple example

The logic **CL**_o.

Take classical logic and add a 'dummy operator' \circ .

More on **CL** in a moment . . .

The Sherlock Holmes Twist

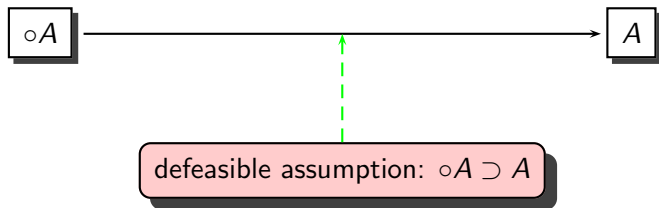
Interpret $\circ A$ by “By the given evidence it is reasonable to assume A ”.

- ▶ If our detective has reason to assume A , — $\circ A$
- ▶ infer that A is the case – defeasibly.

Is \mathbf{CL}_\circ already a good logic for Sherlock Holmes?

- ▶ Suppose he gets some evidence that suggests that A is the case, $\text{---} \circ A$.
- ▶ He cannot infer A yet with \mathbf{CL}_\circ .
- ▶ Option 1: do nothing. This would be a boring detective.
- ▶ Option 2: 'jump to the conclusion A '
- ▶ however, what now if he gets different evidence that indicates that $\neg A$ is the case, $\text{---} \circ \neg A$?

How would Holmes reason?



How to model this formally? \Rightarrow Adaptive Logics

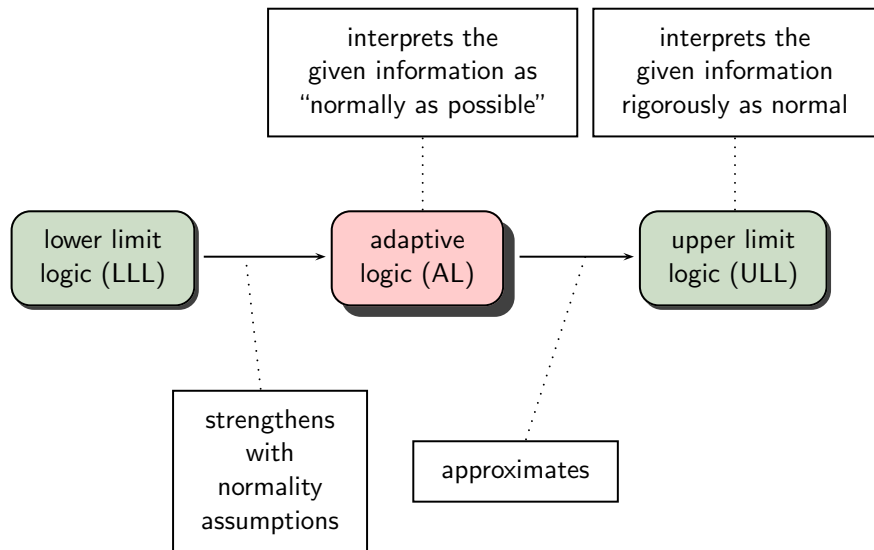
The Three Parts that Characterize Adaptive Logics

1. The Lower Limit Logic

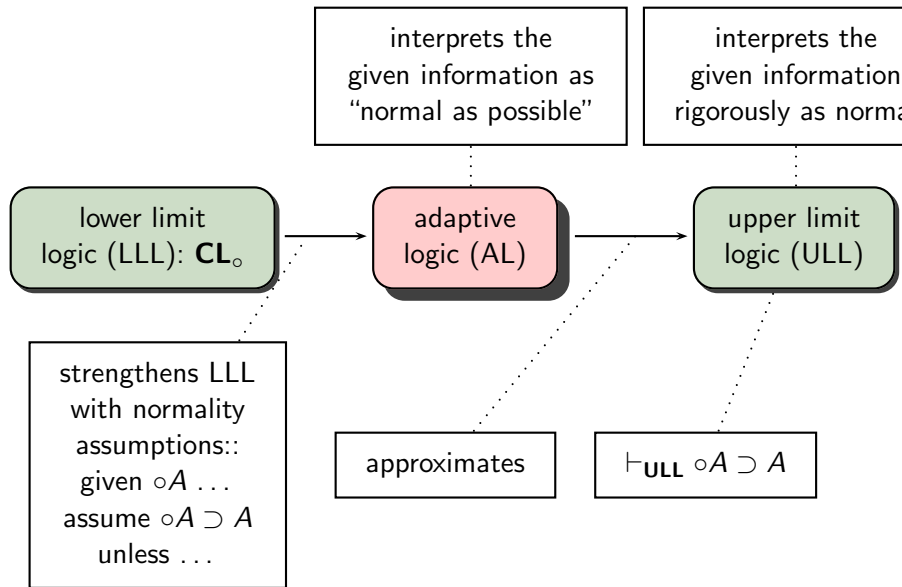
2. The set of abnormalities

3. The adaptive strategy

The Lower Limit Logic



The Lower Limit Logic: in our example



Requirements for Lower Limit Logics

- ▶ reflexive:
- ▶ transitive:
- ▶ monotonic:
- ▶ compact:
- ▶ has a characteristic semantics
- ▶ often we need to speak about an enriched **LLL**: it is enriched by classical operators denoted by a “check”: e.g. $\check{\neg}$, $\check{\forall}$ etc. (we will discuss this topic in more detail later)
- ▶ some papers make the distinction between the enriched **LLL** and **LLL** explicit by writing **LLL**⁺ for the former system.
- ▶ premise sets are considered to not contain “checked connectives”

Inference Rules and Proofs: Hilbert Style

CL is defined by Modus Ponens (MP) and the following axiom schemata:

$$(A \supset 1) \quad A \supset (B \supset A)$$

$$(A \supset 2) \quad (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$$

$$(A \supset 3) \quad ((A \supset B) \supset A) \supset A$$

$$(A \wedge 1)$$

$$(A \wedge 2)$$

$$(A \wedge 3)$$

$$(A \vee 1)$$

Weakening

$$(A \vee 2)$$

Weakening

$$(A \vee 3) \quad (A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C)) \text{ Re. by cases}$$

$$(A \equiv 1) \quad (A \equiv B) \supset (A \supset B)$$

$$(A \equiv 2) \quad (A \equiv B) \supset (B \supset A)$$

$$(A \equiv 3) \quad (A \supset B) \supset ((B \supset A) \supset (A \equiv B))$$

$$(A \neg 1) \quad (A \supset \neg A) \supset \neg A$$

$$(A \neg 2)$$

Ex Contradictione Quodlibet

$$(A \neg 3)$$

Excl. Middle

Example: A proof

Show: $\{p \wedge q, p \supset r, r \supset s\} \vdash s.$

Task: Proof $A \supset A$

Tip: you only need $(A \supset 1)$ and $(A \supset 2)$.

Proof: $(B \vee C) \supset (\neg B \supset C)$

Proof: $A \supset B, B \supset C \vdash A \supset C$

Proof: $\neg\neg A \supset A$

Proof: $A \supset \neg\neg A$

Resolution Theorem

The Resolution Theorem

$\Gamma \vdash A \supset B$ implies $\Gamma \cup \{A\} \vdash B$.

Proof:

The Deduction Theorem

$\Gamma \cup \{A\} \vdash B$ implies $\Gamma \vdash A \supset B$.

Proof.

- ▶ Idea: every proof of B from $\Gamma \cup \{A\}$ can be transformed into a proof of $A \supset B$ from Γ .
- ▶ Let \mathcal{P} be an arbitrary proof of B from $\Gamma \cup \{A\}$.
- ▶ We show by induction on the length of \mathcal{P} that for every line l in proof \mathcal{P} on which C is proved, $A \supset C$ can be proven from Γ .
- ▶ “ $l = 1$ ”: C is either an axiom or a premise in Γ or A .
 - ▶ Suppose C is an axiom. Note that by (A \supset 1), $C \supset (A \supset C)$. Also, we can introduce C since it is an axiom. By MP, $A \supset C$.
 - ▶ Suppose $C \in \Gamma$. By (A \supset 1), $C \supset (A \supset C)$. By MP, $A \supset C$.
 - ▶ Suppose $C = A$. We have shown above that $\vdash A \supset A$.

The Deduction Theorem

$\Gamma \cup \{A\} \vdash B$ implies $\Gamma \vdash A \supset B$.

Proof (continued):

- ▶ “ $l \Rightarrow l+1$ ”: (i) C is either an axiom, or (ii) $C \in \Gamma$, or (iii) $C = A$, or (iv) C is obtained via MP from D and $D \supset C$ which were derived on lines l' and l'' (where $l', l'' \leq l$). The first three cases are as in the induction base. Hence, suppose (iv).
 - ▶ By the induction hypothesis, we have $\Gamma \vdash A \supset D$ and $\Gamma \vdash A \supset (D \supset C)$.
 - ▶ Hence, there are proofs P_1 of $X = A \supset D$ from Γ and P_2 of $Y = A \supset (D \supset C)$ from Γ .
 - ▶ We concatenate P_1 and P_2 obtaining P_3 .
 - ▶ By (A \supset 2), $Z = (A \supset (D \supset C)) \supset ((A \supset D) \supset (A \supset C))$.
 - ▶ By Y, Z and MP, $W = (A \supset D) \supset (A \supset C)$.
 - ▶ By X, W and MP, $A \supset C$.

Semantics: Providing Meaning

- ▶ atomic level: assignment function $v : \mathcal{A} \rightarrow \{0, 1\}$
- ▶ model: M is associated with an assignment function
- ▶ truth in a model defined recursively:
 - ▶ $M \models A$ where $A \in \mathcal{A}$ iff $v(A) = 1$
 - ▶ $M \models A \wedge B$ iff $M \models A$ and $M \models B$
 - ▶ $M \models A \vee B$ iff $M \models A$ or $M \models B$
 - ▶ $M \models \neg A$ iff $M \not\models A$
 - ▶ $M \models A \supset B$ iff $M \not\models A$ or $M \models B$
 - ▶ $M \models A \equiv B$ iff ($M \models A$ iff $M \models B$)
- ▶ or via an evaluation function $v_M : \mathcal{W} \rightarrow \{0, 1\}$
 - ▶ $v_M(A) = v(A)$ where $A \in \mathcal{A}$
 - ▶ $v_M(A \wedge B) = \min(v_M(A), v_M(B))$
 - ▶ $v_M(A \vee B) = \max(v_M(A), v_M(B))$
 - ▶ $v_M(A \supset B) = \max(1 - v_M(A), v_M(B))$
 - ▶ etc.
- ▶ Semantic consequence: $\Gamma \Vdash A$ iff for all models M : if $M \models B$ for all $B \in \Gamma$ then $M \models A$.
- ▶ Truth-functional operator π :
 $v_M(\pi(A_1, \dots, A_n)) = f(v_M(A_1), \dots, v_M(A_n))$ for some function f

Soundness

$\Gamma \vdash D$ implies $\Gamma \Vdash D$.

Proof.

- ▶ Take an arbitrary proof of D from Γ . Let M be an arbitrary model of Γ .
- ▶ We prove by induction over the length of the proof that for each formula E derived at a line l , $M \vDash E$.
- ▶ “ $l = 1$ ”: E is either (i) an axiom or (ii) $E \in \Gamma$. (ii) is trivial. Suppose (i). Suppose $E = A \supset (B \supset A)$ (see $(A \supset 1)$). Let M be a model of Γ . $M \vDash A \supset (B \supset A)$ iff $v_M(A \supset (B \supset A)) = 1$ iff $\max(v_M(1 - v_M(A), \max(1 - v_M(B), v_M(A)))) = 1$. Note that the latter holds. The proof is similar for the other axioms.
- ▶ “ $l \Rightarrow l+1$ ”: Either (i) E is an axiom, or (ii) $E \in \Gamma$ or (iii) E is derived via MP from $F \supset E$ and F where $F \supset E$ and F are derived at lines $l' \leq l$ and $l'' \leq l$ resp. Only (iii) is non-trivial. By the induction hypothesis, $M \vDash F$ and $M \vDash F \supset E$. Thus, $v_M(F) = v_M(F \supset E) = \max(1 - v_M(F), v_M(E)) = 1$. Thus, $M \vDash E$.

Completeness

$\Gamma \Vdash A$ implies $\Gamma \vdash A$.

Proof: we need some preparation for that.

- ▶ A set Γ is inconsistent iff $\Gamma \vdash A$ for all $A \in \mathcal{W}$.
- ▶ Γ is consistent iff Γ is not inconsistent.

Proposition 1

If $\Gamma \not\vdash A$ then $\Gamma \cup \{\neg A\}$ is consistent.

Proof.

- ▶ Suppose $\Gamma \cup \{\neg A\}$ is inconsistent.
- ▶ Hence, $\Gamma \cup \{\neg A\} \vdash \neg\neg A$.
- ▶ Hence, by the deduction theorem, $\Gamma \vdash \neg A \supset \neg\neg A$.
- ▶ By $(A\neg 1)$ and MP, $\Gamma \vdash \neg\neg A$.
- ▶ Since $\neg\neg A \vdash A$ (see above), $\Gamma \vdash A$.



Explosion in Hilbert

Show $\{A, \neg A\} \vdash B$. Tip: use $(A\neg 2)$ and MP.

Proposition 2

If Γ is consistent then there is a model of Γ .

Proof.

- ▶ Let \mathcal{W} be enumerated by A_1, A_2, \dots
- ▶ Let $\Gamma_0 = \Gamma$ and define

$$\Gamma_{i+1} = \begin{cases} \Gamma_i \cup \{A_i\} & \text{if } \Gamma_i \cup \{A_i\} \text{ is consistent} \\ \Gamma_i \cup \{\neg A_i\} & \text{else.} \end{cases}$$

- ▶ Let $\Gamma^* = \bigcup_{i \geq 0} \Gamma_i$.
- ▶ Claim: Γ^* is consistent. Proof: by induction. IB: Γ_0 is consistent by supposition. “ $i \rightarrow i+1$ ”: by definition of Γ_{i+1} . Hence, each Γ_i is consistent. Assume Γ^* is inconsistent. Hence, $\Gamma^* \vdash A_1$ and $\Gamma^* \vdash \neg A_1$. Hence (by compactness), there is a Γ_i such that $\Gamma_i \vdash A_1$ and there is a Γ_j such that $\Gamma_j \vdash \neg A_1$. Take $k = \max(i, j)$. Then (by monotonicity), $\Gamma_k \vdash A_1$ and $\Gamma_k \vdash \neg A_1$. Hence, Γ_k is inconsistent (since $A, \neg A \vdash B$).



- ▶ Claim: $\Gamma^* \vdash A$ implies $A \in \Gamma^*$ (*Deductive Closure*). Suppose $\Gamma^* \vdash A$ and assume $A \notin \Gamma^*$. Hence, $\neg A \in \Gamma^*$. But then $\Gamma^* \vdash A$ and $\Gamma^* \vdash \neg A$ and hence Γ^* is inconsistent,—a contradiction.
- ▶ Claim: $B, C \in \Gamma^*$ iff $B \wedge C \in \Gamma^*$. Suppose $B, C \in \Gamma^*$. Hence, by ResThm and $(A \wedge 3)$, $\Gamma^* \vdash B \wedge C$. Hence, $B \wedge C \in \Gamma^*$. Suppose $B \wedge C \in \Gamma^*$. Hence, $\Gamma^* \vdash B$ (by $(A \wedge 1)$). Thus $B \in \Gamma^*$.
- ▶ Claim: $B \vee C \in \Gamma^*$ iff ($B \in \Gamma^*$ or $C \in \Gamma^*$). Let $B \vee C \in \Gamma^*$. Since $B \vee C \vdash \neg B \supset C$, $\neg B \supset C \in \Gamma^*$. Suppose $B \notin \Gamma^*$ and hence $\neg B \in \Gamma^*$. Hence, by MP, $C \in \Gamma^*$. Now suppose $B \in \Gamma^*$. Since $B \supset (B \vee C)$ also $B \vee C \in \Gamma^*$. The case for $C \in \Gamma^*$ is analogous.
- ▶ Claim: $A \in \Gamma^*$ iff $\neg A \notin \Gamma^*$. Suppose $A \in \Gamma^*$. Assume $\neg A \in \Gamma^*$, then Γ^* is inconsistent,—a contradiction. The other way is analogous.

- ▶ Define M via the assignment
- ▶ We show by induction over the length of a formula that $M \models A$ iff $A \in \Gamma^*$.
 - ▶ $A \in \mathcal{A}$: by definition.
 - ▶ Let B, C be such that $M \models B [C]$ iff $B [C] \in \Gamma^*$.
 - ▶ Let $A = B \wedge C$. Let $B \wedge C \in \Gamma^*$. Hence, $B, C \in \Gamma^*$. Hence, by the induction hypothesis, $M \models B$ and $M \models C$. Hence, $M \models B \wedge C$. The other way around is analogous.
 - ▶ Let $A = B \vee C$. Let $B \vee C \in \Gamma^*$. Hence, $B \in \Gamma^*$ or $C \in \Gamma^*$. By the induction hypothesis, $M \models B$ or $M \models C$. Hence, $M \models B \vee C$. The other way around is analogous.
 - ▶ Let $A = \neg B$. Let $\neg B \in \Gamma^*$. Hence, $B \notin \Gamma^*$. By the induction hypothesis, $M \not\models B$ and hence $M \models \neg B$. The other way around is analogous.
 - ▶ etc. □

Completeness: $\Gamma \models A$ implies $\Gamma \vdash A$.

Proof.



Abnormalities

- ▶ They determine the normality assumptions by means of which the AL strengthens the LLL.
- ▶ In other words, they determine what it means to interpret premises “normal”. (we will come to the “as possible part” later)
- ▶ characterized by a logical form F
- ▶ the set of all abnormalities denoted by Ω

...in our example ...

- ▶ recall: the normality assumption was that if $\circ A$ then A is the case
- ▶ hence, $\Omega = \{\circ A \wedge \neg A\}$

3. The adaptive strategy

effects both,

1. the proof theory, and
2. the semantics

Let's start with the proof theory.

Adaptive proofs

1 $P_1 \dots$; PREM \emptyset

\vdots \vdots \vdots

n $P_n \dots$; PREM \emptyset

\vdots \vdots \vdots

l_1 $A_1 \dots$; RU Δ_1

\vdots \vdots \vdots

l $A \dots$; RC $\Delta_1 \cup \Delta_2 \cup \Delta_3$

1. Line
number

2. formula

3. justification

4. **condition**

Premises

Ceteris
normalibus

support

Conclusion

Adaptive proofs: the generic rules

- ▶ **PREM**ises are introduced on the empty condition (no normality assumption is needed for that)

$$\text{If } A \in \Gamma : \quad \frac{\begin{array}{c} \vdots \\ A \end{array} \quad \begin{array}{c} \vdots \\ \emptyset \end{array}}{\quad} \quad (\text{PREM})$$

- ▶ the **U**nconditional **R**ule:

$$\text{If } A_1, \dots, A_n \vdash_{\text{LLL}} B : \quad \frac{\begin{array}{c} A_1 \quad \Delta_1 \\ \vdots \quad \vdots \\ A_n \quad \Delta_n \end{array}}{\begin{array}{c} B \quad \Delta_1 \cup \dots \cup \Delta_n \end{array}} \quad (\text{RU})$$

These two rules give us the full power of the LLL:

If $\Gamma \vdash_{\text{LLL}} A$, then $\Gamma \vdash_{\text{AL}} A$.

Adaptive proofs: the conditional rule

$$\text{If } A_1, \dots, A_n \vdash_{\text{LLL}} B \checkmark \text{Dab}(\Theta) : \frac{\begin{array}{c} A_1 \quad \Delta_1 \\ \vdots \quad \vdots \\ A_n \quad \Delta_n \\ \hline B \quad \Delta_1 \cup \dots \cup \Delta_n \cup \Theta \end{array}}{\text{(RC)}}$$

- ▶ $\text{Dab}(\Theta)$ is a notational convention that denotes the disjunction of abnormalities in Θ ,
- ▶ where $\Theta \subseteq \Omega$ is a finite set of abnormalities
- ▶ as in RU, the conditions of the used lines $(\Delta_1, \dots, \Delta_n)$ are carried forward

The rationale of RC:

From A_1, \dots, A_n follows by the LLL that either B is true or one of the abnormalities in Θ is true. The AL allows us to conditionally derive B under the assumption that neither of the abnormalities in Θ is true.

Time for examples ...

- ▶ recall: $\circ A \vdash_{\mathbf{CL}_\circ} A \vee (\circ A \wedge \neg A)$
- ▶ a conditional derivation by means of RC:

1	$\circ A$	PREM	\emptyset
2	A	1;RC	$\{\circ A \wedge \neg A\}$

- ▶ also: $\{\circ A, A \supset B\} \vdash_{\mathbf{CL}_\circ} B \vee (\circ A \wedge \neg A)$

3	$A \supset B$	PREM	\emptyset
4	B	1,3;RC	$\{\circ A \wedge \neg A\}$

Now what if we also have $\circ \neg A$?

Marking of lines in adaptive proofs

Example 1: a simple case of marking

1	$\circ A$	PREM	\emptyset
⁶ 2	A	1;RC	$\{\circ A \wedge \neg A\}$
3	$A \supset B$	PREM	\emptyset
⁶ 4	B	1,3;RC	$\{\circ A \wedge \neg A\}$
5	$\neg A$	PREM	\emptyset
6	$\circ A \wedge \neg A$	1,5;RU	\emptyset

- ▶ the condition has been derived (on the empty condition)
- ▶ so –obviously– it's not save anymore to assume that $\circ A \wedge \neg A$ is not the case (resp. that $\circ A$ implies A)
- ▶ thus, mark all lines with this assumption

Marking of lines in adaptive proofs

Example 2: a more complex case of marking

1	$\circ A$	PREM	\emptyset
⁷ 2	A	1;RC	$\{\circ A \wedge \neg A\}$
3	$A \supset B$	PREM	\emptyset
⁷ 4	B	1,3;RC	$\{\circ A \wedge \neg A\}$
5	$\neg A \vee \neg B$	PREM	\emptyset
6	$\circ B$	PREM	\emptyset
7	$(\circ A \wedge \neg A) \vee (\circ B \wedge \neg B)$	1,5,6;RU	\emptyset

- ▶ here $\circ A \wedge \neg A$ is part of a *disjunction of abnormalities* that has been derived on the empty condition (line 7)
- ▶ note that the formula at line 7 is a Dab-formula
- ▶ this disjunction is *minimal*: right now we have no means to decide whether $\circ A \wedge \neg A$ or $\circ B \wedge \neg B$ is the case (or even both)
- ▶ hence, we're cautious and mark lines that intersect with members of the minimal disjunction of abnormalities on line 7

Adaptive strategies and marking

- ▶ the specifics of the marking definition of an AL depend on the adaptive strategy that is used
- ▶ there are two standard strategies
 1. **reliability strategy**
 2. **minimal abnormality strategy**
- ▶ we write **AL^r** for an AL characterized by a triple $\langle \mathbf{LLL}, \Omega, \mathbf{reliability} \rangle$
- ▶ we write **AL^m** for an AL characterized by a triple $\langle \mathbf{LLL}, \Omega, \mathbf{minimal\ abnormality} \rangle$

The reliability strategy: marking

- ▶ a **stage** of a proof is a list of consecutive lines
- ▶ where $\text{Dab}(\Delta_1), \text{Dab}(\Delta_2), \dots$ are the minimal disjunctions of abnormalities that are derived at some stage s on the empty condition from the premise set Γ , let $\Sigma_s(\Gamma) =_{\text{df}} \{\Delta_1, \Delta_2, \dots\}$
- ▶ the set of **unreliable formulas at stage s** is defined by
$$U_s(\Gamma) =_{\text{df}} \Delta_1 \cup \Delta_2 \cup \dots = \bigcup \Sigma_s(\Gamma)$$

Marking definition for the reliability strategy

A line l with condition Δ is marked at stage s iff $\Delta \cap U_s(\Gamma) \neq \emptyset$.

- ▶ *in words*: a line is marked iff its condition contains unreliable formulas.
- ▶ *put differently*: a line is marked if its condition contains formulas that are part of minimal disjunctions of abnormalities
- ▶ lets take a look at our examples ...

Example 1: a simple case of marking

1	$\circ A$	PREM	\emptyset
⁶ 2	A	1;RC	$\{\circ A \wedge \neg A\}$
3	$A \supset B$	PREM	\emptyset
⁶ 4	B	1,3;RC	$\{\circ A \wedge \neg A\}$
5	$\neg A$	PREM	\emptyset
6	$\circ A \wedge \neg A$	1,5;RU	\emptyset

► $\Sigma_6(\Gamma) =$

► $U_6(\Gamma) =$

Example 2: a more complex case of marking

1	$\circ A$	PREM	\emptyset
7 2	A	1;RC	$\{\circ A \wedge \neg A\}$
3	$A \supset B$	PREM	\emptyset
7 4	B	1,3;RC	$\{\circ A \wedge \neg A\}$
5	$\neg A \vee \neg B$	PREM	\emptyset
6	$\circ B$	PREM	\emptyset
7	$(\circ A \wedge \neg A) \vee (\circ B \wedge \neg B)$	1,5,6;RU	\emptyset

► $\Sigma_7(\Gamma) =$

► $U_7(\Gamma) =$

1	$\circ A$	PREM	\emptyset
2	A	1;RC	$\{\circ A \wedge \neg A\}$
3	$A \supset B$	PREM	\emptyset
4	B	1,3;RC	$\{\circ A \wedge \neg A\}$
5	$\neg A \vee \neg B$	PREM	\emptyset
6	$\circ B$	PREM	\emptyset
7	$(\circ A \wedge \neg A) \vee (\circ B \wedge \neg B)$	1,5,6;RU	\emptyset
8	$\neg B$	PREM	\emptyset
9	$\circ B \wedge \neg B$	6,8;RU	\emptyset

- ▶ $\Sigma_7(\Gamma) = \{ \{ (\circ A \wedge \neg A), (\circ B \wedge \neg B) \} \}$
- ▶ $U_7(\Gamma) = \{ \circ A \wedge \neg A, \circ B \wedge \neg B \}$
- ▶ note that at line 9 the formula at stage 7 loses its status of being a minimal(!) Dab-formula
- ▶ $\Sigma_9(\Gamma) =$
- ▶ $U_9(\Gamma) =$

Markings come and go: lines which are unmarked may be marked at a later stage, and be unmarked again at an even later stage.

BUT: when does a formula count as a consequence of the AL?

Final derivability

A formula A is finally derived at line l at a finite stage s iff (i) l is unmarked at stage s and (ii) for every extension of the proof in which l is marked, there is a further extension in which l is unmarked.

(Note: the extensions in question may be infinite.)

The adaptive derivability relation $\vdash_{\mathbf{AL}}$ and the adaptive consequence set $Cn_{\mathbf{AL}}$

$\Gamma \vdash_{\mathbf{AL}} A$ iff there is an **AL**-proof from Γ in which A is finally derived.

$$Cn_{\mathbf{AL}}(\Gamma) = \{A \mid \Gamma \vdash_{\mathbf{AL}} A\}$$

Some nice properties of the consequence/derivability relation

- ▶ recall: $U_s(\Gamma) =_{\text{df}} \bigcup \Sigma_s(\Gamma)$ where $\Sigma_s(\Gamma) =_{\text{df}} \{\Delta \mid \text{Dab}(\Delta) \text{ is a minimal Dab-formula at stage } s\}$
- ▶ let $\Sigma(\Gamma)$ be the set of all Δ for which $\Gamma \vdash_{\text{LLL}} \text{Dab}(\Delta)$ and for all $\Delta' \subset \Delta$, $\Gamma \not\vdash_{\text{LLL}} \text{Dab}(\Delta')$.
- ▶ let $U(\Gamma) =_{\text{df}} \bigcup \Sigma(\Gamma)$

Theorem 1

$\Gamma \vdash_{\text{ALr}} A$ iff there is a $\Delta \subseteq \Omega$ for which $\Gamma \vdash_{\text{LLL}} A \checkmark \text{Dab}(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$.

Another example

Suppose a reliable although not infallible witness report that

- ▶ Mr. X wore a long black coat in the bar in which he was seen half an hour before the murder. — o/l

Another reliable although not infallible source however witnesses that

- ▶ Mr. X wore a short dark blue jacket and black trousers at the same time. — o/j

Obviously $\neg(l \wedge j)$, since both cannot be the case. Moreover, we have

- ▶ If Mr. X was dressed in a long black coat, then he was dressed in a dark way. — $l \supset m$
- ▶ If Mr. X was dressed in a short dark blue jacket and black trousers, then he was dressed in a dark way. — $j \supset m$

1	$\circ l$	PREM	\emptyset
2	$\circ j$	PREM	\emptyset
3	$\neg(l \wedge j)$	PREM	\emptyset
4	$l \supset m$	PREM	\emptyset
5	$j \supset m$	PREM	\emptyset
¹⁰ 6	l	1; RC	$\{\circ l \wedge \neg l\}$
?7	m	4, 6; RU	$\{\circ l \wedge \neg l\}$
¹⁰ 8	j	2; RC	$\{\circ j \wedge \neg j\}$
?9	m	5, 8; RU	$\{\circ j \wedge \neg j\}$
10	$(\circ l \wedge \neg l) \vee (\circ j \wedge \neg j)$	1,2,3; RU	\emptyset

- ▶ according to the reliability strategy lines 7 and 9 are marked
- ▶ the rationale is: since $(\circ l \wedge \neg l) \vee (\circ j \wedge \neg j)$ is a minimal Dab-formula, one of the two abnormalities is the case **or even both**
- ▶ in case both are the case, neither of the lines 7 and 9 is safe

1	o/l	PREM	\emptyset
2	o/j	PREM	\emptyset
3	$\neg(l \wedge j)$	PREM	\emptyset
4	$l \supset m$	PREM	\emptyset
5	$j \supset m$	PREM	\emptyset
¹⁰ 6	l	1; RC	$\{o/l \wedge \neg/l\}$
[?] 7	m	4, 6; RU	$\{o/l \wedge \neg/l\}$
¹⁰ 8	j	2; RC	$\{o/j \wedge \neg/j\}$
[?] 9	m	5, 8; RU	$\{o/j \wedge \neg/j\}$
10	$(o/l \wedge \neg/l) \vee (o/j \wedge \neg/j)$	1,2,3; RU	\emptyset

- ▶ another rationale: interpreting the premises as normally as possible means that we assume that as less abnormalities as possible are the case
- ▶ for the disjunction at line 10 this means that we assume that only one of the two abnormalities is the case (we don't know which one though)
- ▶ however, then at least one of the two assumptions at line 7 and 9 can be considered as safe and thus it is still (defeasibly) warranted to infer m

The minimal abnormality strategy

- ▶ Recall: $\Sigma_s(\Gamma) =_{\text{df}} \{\Delta \mid \text{Dab}(\Delta) \text{ is a minimal Dab-formula at stage } s\}$.
- ▶ $\Phi_s(\Gamma)$ is the set of all minimal choice sets of $\Sigma_s(\Gamma)$
- ▶ a *choice set* of $\{\Delta_i \mid i \in I\}$ is a set that contains a member of each Δ_i ($i \in I$)
- ▶ φ is a *minimal choice set* of iff there is no choice set φ' such that $\varphi' \subset \varphi$
- ▶ example: Let $S = \{\{1, 2\}, \{2, 3\}\}$.
 - ▶ $\{1\}$ is not a choice set of S since $\{1\} \cap \{2, 3\} = \emptyset$
 - ▶ $\{1, 2\}$ is a choice set of S
 - ▶ $\{1, 3\}$ and $\{2\}$ are the minimal choice sets of S
- ▶ each set $\varphi \in \Phi_s(\Gamma)$ offers a *minimally abnormal* interpretation of the given premises resp. minimal Dab-formulas according to the current stage of the proof s . By *minimally abnormal* we mean that as few abnormalities as possible are interpreted as true.

Marking for minimal abnormality

A line l with conditions Δ and formula A is marked at stage s iff

- (i) there is no $\varphi \in \Phi_s(\Gamma)$ such that $\varphi \cap \Delta = \emptyset$ or
- (ii) for some $\varphi \in \Phi_s(\Gamma)$ there is no line at which A is derived on a condition Θ for which $\Theta \cap \varphi = \emptyset$.

what does this mean, intuitively ...

- ▶ condition (i) expresses that the assumption Δ on which A is derived is not warranted in any minimally abnormal interpretation offered by $\Phi_s(\Gamma)$, since in each $\varphi \in \Phi_s(\Gamma)$ there is an abnormality that is also in Δ and since the assumption expressed by the condition Δ is that no abnormality in Δ is true.
- ▶ condition (ii) expresses that there is a minimally abnormal interpretation $\varphi \in \Phi_s(\Gamma)$ such that A is not derived under any condition that is warranted in φ .

So, when is a line unmarked according to minimal abnormality?

A line l with formula A and condition Δ is not marked at stage s iff

- (i) there is a $\varphi \in \Phi_s(\Gamma)$ such that $\Delta \cap \varphi = \emptyset$ and
- (ii) for each $\varphi \in \Phi_s(\Gamma)$ there is a Δ_φ such that $\Delta_\varphi \cap \varphi = \emptyset$ and A is derived on the condition Δ_φ at stage s .

What does this mean, intuitively ...

- ▶ condition (i) expresses that there is a minimally abnormal interpretation $\varphi \in \Phi_s(\Gamma)$ in which the assumption Δ is warranted
- ▶ condition (ii) expresses that for each minimally abnormal interpretation $\varphi \in \Phi_s(\Gamma)$ our A is derived on an assumption Δ_φ that is warranted in φ

1	o/l	PREM	\emptyset
2	o/j	PREM	\emptyset
3	$\neg(l \wedge j)$	PREM	\emptyset
4	$l \supset m$	PREM	\emptyset
5	$j \supset m$	PREM	\emptyset
¹⁰ 6	l	1; RC	$\{o/l \wedge \neg l\}$
7	m	4, 6; RU	$\{o/l \wedge \neg l\}$
¹⁰ 8	j	2; RC	$\{o/j \wedge \neg j\}$
9	m	5, 8; RU	$\{o/j \wedge \neg j\}$
10	$(o/l \wedge \neg l) \vee (o/j \wedge \neg j)$	1,2,3; RU	\emptyset

- ▶ $\Sigma_{10}(\Gamma) = \{ \{ (o/l \wedge \neg l), (o/j \wedge \neg j) \} \}$
- ▶ $\Phi_{10}(\Gamma) = \{ \{ o/l \wedge \neg l \}, \{ o/j \wedge \neg j \} \}$
- ▶ lines 6 and 8 are marked since they violate condition (ii)
- ▶ lines 7 and 9 are not marked:
 - ▶ concerning (i): there is a minimal choice set with which the condition has empty intersection
 - ▶ concerning (ii): there is no choice set that intersects with both conditions, $\{o/l \wedge \neg l\}$ and $\{o/j \wedge \neg j\}$

Floating conclusions and the adaptive strategies

Floating conclusion

A is a *floating conclusion* in case it is reached by various conflicting arguments.

- ▶ reliability blocks the floating conclusion m from $\Gamma = \{\circ l, \circ j, \neg(l \wedge j), l \supset m, j \supset m\}$:

$$\Gamma \not\vdash_{\text{CL}_\circ} m$$

- ▶ minimal abnormalities derives the floating conclusion:

$$\Gamma \vdash_{\text{CL}_\circ^m} m$$

Some nice property

- ▶ recall: $\Phi_s(\Gamma)$ is the set of minimal choice sets of $\Sigma_s(\Gamma)$
- ▶ $\Sigma(\Gamma)$ is the set of all Δ for which $\Gamma \vdash_{\text{LLL}} \text{Dab}(\Delta)$ and for all $\Delta' \subset \Delta$, $\Gamma \not\vdash_{\text{LLL}} \text{Dab}(\Delta')$.
- ▶ Let $\Phi(\Gamma)$ be the set of minimal choice sets of $\Sigma(\Gamma)$

Theorem 2

$\Gamma \vdash_{\text{AL}^m} A$ iff for every $\varphi \in \Phi(\Gamma)$ there is a $\Delta \subseteq \Omega$ for which $\Delta \cap \varphi = \emptyset$ and $\Gamma \vdash_{\text{LLL}} A \checkmark \text{Dab}(\Delta)$.

Final derivability revisited?

- ▶ recall: A formula A is finally derived at a finite stage s at line l iff (i) l is unmarked at stage s and (ii) for every extension of the proof in which l is marked, there is a further extension in which l is unmarked.
- ▶ say: A formula A is *finally *-derived* at a finite stage s at line l iff (i) l is unmarked at stage s and (ii) for every *finite* extension of the proof in which l is marked there is a *finite* further extension in which l is not marked.
- ▶ Let $\Gamma \vdash_{\mathbf{AL}}^* A$ iff there is a proof in which A is finally *-derived.

Does this work?

$\Gamma \vdash_{\mathbf{AL}} A$ iff $\Gamma \vdash_{\mathbf{AL}}^* A$.

Nope

E.g.

$\Gamma = \{(\circ A_i \wedge \neg A_i) \vee (\circ A_j \wedge \neg A_j) \mid j > i > 0\} \cup \{B \vee A_i \mid i > 1\}$.

Here $\Gamma \not\vdash_{\mathbf{AL}} B$ while $\Gamma \vdash_{\mathbf{AL}}^* B$.



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In Cédric Dégremon, Laurent Keiff, and Helge Rückert, editors, *Dialogues, Logics and Other Strange Things. Essays in Honour of Shahid Rahman*, pages 27–51. College Publications, London, 2009.



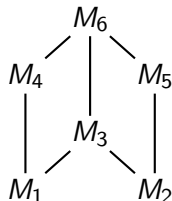
Peter Verdée.

Adaptive logics using the minimal abnormality strategy are π_1^1 -complex.

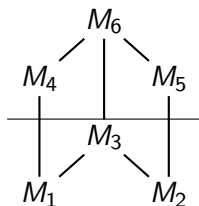
Synthese, 167:93–104, 2009.

Semantics for Adaptive Logics: The basic idea

- ▶ Take the set of **LLL**-models of a premise set Γ
- ▶ order them according to their **abnormal part**, i.e.
$$\text{Ab}(M) = \{A \in \Omega \mid M \models A\}$$



- ▶ in flat adaptive logics in standard format this is done by means of: $M_1 \prec M_2$ iff $\text{Ab}(M_1) \subset \text{Ab}(M_2)$
- ▶ select models that are beyond a certain threshold

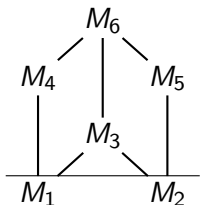


What threshold?

the threshold depends on the strategy:

Minimal Abnormality

- ▶ Idea: take the *minimally abnormal* models
- ▶ $M \in \mathcal{M}_{\mathbf{AL}^m}(\Gamma)$ iff $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ and for all $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$, if $\text{Ab}(M') \subseteq \text{Ab}(M)$ then $\text{Ab}(M') = \text{Ab}(M)$.

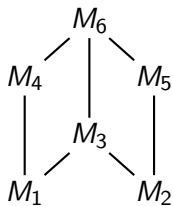


Reliability

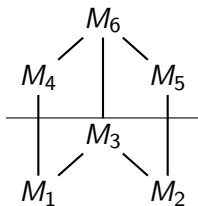
- ▶ Idea: take models whose abnormal part only consists of unreliable abnormalities
- ▶ we call these models “reliable”
- ▶ $M \in \mathcal{M}_{\mathbf{AL}^r}(\Gamma)$ iff $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ and $\text{Ab}(M) \subseteq U(\Gamma)$

Let's go back to our example...

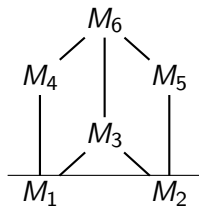
- ▶ $\Gamma = \{\diamond l, \diamond j, \neg(l \wedge j), l \supset m, j \supset m\}$
- ▶ $\Gamma \vdash_{\mathbf{T}} (\circ l \wedge \neg l) \vee (\circ j \wedge \neg j)$
- ▶ $\Gamma \not\vdash_{\mathbf{T}} \circ l \wedge \neg l,$
- ▶ $\Gamma \not\vdash_{\mathbf{T}} \circ j \wedge \neg j$
- ▶ hence, $U(\Gamma) = \{\circ l \wedge \neg l, \circ j \wedge \neg j\}$
- ▶ we have for instance the following models M_1, \dots, M_6 where
 - ▶ $\text{Ab}(M_1) = \{\circ l \wedge \neg l\},$
 - ▶ $\text{Ab}(M_2) = \{\circ j \wedge \neg j\},$
 - ▶ $\text{Ab}(M_3) = \{\circ l \wedge \neg l, \circ j \wedge \neg j\},$
 - ▶ $\text{Ab}(M_4) = \{\circ l \wedge \neg l, \circ k \wedge \neg k\}$
 - ▶ $\text{Ab}(M_5) = \{\circ j \wedge \neg j, \circ o \wedge \neg o\}$
 - ▶ $\text{Ab}(M_6) = \{\circ l \wedge \neg l, \circ j \wedge \neg j, \circ k \wedge \neg k, \circ o \wedge \neg o\}$
- ▶ models M_1 and M_2 are minimally abnormal
- ▶ models $M_1, M_2,$ and M_3 are reliable



(a)



(b)



(c)

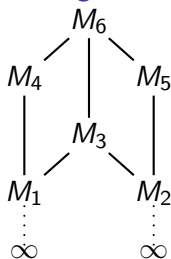
- (a) the ordering of the models according to their abnormal part
- (b) the threshold for the reliable models
- (c) the threshold for the minimal abnormal models

Note that every reliable model is minimal abnormal:

$$\mathcal{M}_{\mathbf{AL}^m}(\Gamma) \subseteq \mathcal{M}_{\mathbf{AL}^r}(\Gamma)$$

Is the ordering of models smooth?

The danger: infinite descending chains without minima



- ▶ w.r.t. the infinite chains without minima there are no minimally abnormal models
- ▶ e.g. if there are only infinite chains without minima there are no minimally abnormal models: $\Gamma \models_{\mathbf{AL}} \perp$ (although there are **LLL**-models of Γ and hence $\Gamma \not\models_{\mathbf{LLL}} \perp$)

Smoothness and Reassurance

- ▶ a partial order $\langle X, \prec \rangle$ is *well-founded* iff there are no infinitely descending chains.
- ▶ a partial order $\langle X, \prec \rangle$ is *smooth* (resp. stoppered) iff for each $x \in X$ there is a minimal element $y \in x$ such that $y \prec x$ or $y = x$
- ▶ what we need is: $\langle \{Ab(M) \mid M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)\}, \subset \rangle$ is smooth. (Note it may be smooth but not well-founded (e.g. invert the order on the natural numbers))

Theorem 3

1. For every **LLL**-model M of Γ , M is minimally abnormal or there is an **LLL**-model M' of Γ such that $Ab(M') \subset Ab(M)$ and M' is minimally abnormal.
2. $\langle \{Ab(M) \mid M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)\}, \subset \rangle$ is smooth.
3. If Γ has **LLL**-models, then there are minimally abnormal models of Γ .
4. If $\Gamma \not\equiv_{\mathbf{LLL}} \perp$ then $\Gamma \not\equiv_{\mathbf{AL}} \perp$.

Simple facts about choice sets

Let in the following $\Sigma = \Sigma(\Gamma)$.

Fact 4

Where φ is a choice set of Σ and $A \in \varphi$: If A satisfies

$$\text{there is a } \Delta \in \Sigma : \varphi \cap \Delta = \{A\} \quad (\dagger)$$

then $\varphi \setminus \{A\}$ is not a choice set of Σ .

Fact 5

Where φ is a choice set of Σ and $A \in \varphi$: If A doesn't satisfy (\dagger) then $\varphi \setminus \{A\}$ is also a choice set of Σ .

Fact 6

Where φ is a choice set of Σ : each $A \in \varphi$ satisfies (\dagger) iff φ is a minimal choice set of Σ .

Lemma 7

Where $\varphi = \{A_1, A_2, \dots\}$ is a choice set of Σ let $\hat{\varphi} = \bigcap_{i \in \mathbb{N}} \varphi_i$ where $\varphi_1 = \varphi$ and

$$\varphi_{i+1} = \begin{cases} \varphi_i & \text{if there is a } \Delta \in \Sigma \text{ s.t. } \varphi_i \cap \Delta = \{A_i\} \\ \varphi_i \setminus \{A_i\} & \text{else} \end{cases}$$

$\hat{\varphi}$ is a minimal choice set of Σ .

Proof.

- ▶ note that φ_i is a choice set of Σ for each $i \in \mathbb{N}$
- ▶ Assume for some $\Delta \in \Sigma$, $\hat{\varphi} \cap \Delta = \emptyset$. Note that since Δ is finite $\Delta \cap \varphi_1 = \{B_1, \dots, B_n\}$ for some $n \in \mathbb{N}$. Assume there no B_j s.t. for all $i \in \mathbb{N}$, $B_j \in \varphi_i \cap \Delta$. Hence, for all B_j 's there is a i_j such that $\varphi_{i_j} \cap \Delta = \emptyset$. Take $k = \max(\{i_j \mid 1 \leq j \leq n\})$, then $B_j \notin \varphi_k \cap \Delta$ since $(\star) \{B_1, \dots, B_n\} \supseteq \varphi_i \cap \Delta \supseteq \varphi_{i+1} \cap \Delta$. This is a contradiction since φ_k is a choice set of Σ and (\star) .
- ▶ Suppose some $A_i \in \hat{\varphi}$ does not satisfy (\dagger) . Hence, for all $\Delta \in \Sigma$, $\hat{\varphi} \cap \Delta \neq \{A_i\}$. Hence, $\varphi_i \cap \Delta \neq \{A_i\}$ for all $\Delta \in \Sigma$. But then $A_i \notin \hat{\varphi}$,—a contradiction.
- ▶ Hence, by the fact above, $\hat{\varphi}$ is a minimal choice set. □

Simple facts about the relation between choice sets and the abnormal parts of models

Lemma 8

If $\varphi \in \Phi(\Gamma)$ then there is a $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ for which $Ab(M) \subseteq \varphi$.

Proof.

- ▶ assume $\nexists M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ s.t. $Ab(M) \subseteq \varphi$
- ▶ then $\Gamma \cup (\Omega \setminus \varphi)^\top$ has no **LLL**-models (where $\Theta^\top =_{\text{df}} \{\neg A \mid A \in \Theta\}$)
- ▶ by the compactness of **LLL**, there is a finite $\Delta \subseteq \Omega \setminus \varphi$ such that $\Gamma \cup \Delta^\top$ has no **LLL**-model
- ▶ hence $\Gamma \Vdash_{\mathbf{LLL}} \text{Dab}(\Delta)$ and hence $\Gamma \vdash_{\mathbf{LLL}} \text{Dab}(\Delta)$
- ▶ this is a contradiction to $\varphi \in \Phi(\Gamma)$ □

Lemma 9

Where $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$, $Ab(M)$ is a choice set of $\Sigma(\Gamma)$.

Proof.

Let $\Delta \in \Sigma(\Gamma)$, then $\Gamma \vdash_{\mathbf{LLL}} \text{Dab}(\Delta)$. Hence, $Ab(M) \cap \Delta \neq \emptyset$. □

Corollary 10

Where $M \in \mathcal{M}_{\text{LLL}}(\Gamma)$, $\text{Ab}(M) \not\subseteq \varphi$ for all $\varphi \in \Phi(\Gamma)$.

Corollary 11

For all $\varphi \in \Phi(\Gamma)$ there is a $M \in \mathcal{M}_{\text{LLL}}(\Gamma)$ such that (i) $\text{Ab}(M) = \varphi$ and (ii) $M \in \mathcal{M}_{\text{AL}^m}(\Gamma)$.

Corollary 12

Where $M \in \mathcal{M}_{\text{LLL}}(\Gamma)$, $M \in \mathcal{M}_{\text{AL}^m}(\Gamma)$ iff $\text{Ab}(M) \in \Phi(\Gamma)$.

Corollary 13 (Strong Reassurance)

For each $M \in \mathcal{M}_{\text{LLL}}(\Gamma)$ there is a $M' \in \mathcal{M}_{\text{AL}^m}(\Gamma)$ such that $\text{Ab}(M') \subseteq \text{Ab}(M)$.

Proof.

By Lemma 9 and Lemma 7 there is a $\varphi \in \Phi(\Gamma)$ such that $\varphi \subseteq \text{Ab}(M)$. By Corollary 11 there is a $M' \in \mathcal{M}_{\text{AL}^m}(\Gamma)$ for which $\text{Ab}(M') = \varphi$. □

Links between the marking and the semantic selection: Reliability

Syntax

Theorem 14

$\Gamma \vdash_{\mathbf{AL}^r} A$ iff there is a $\Delta \subseteq \Omega$ for which $\Gamma \vdash_{\mathbf{LLL}} A \vee \text{Dab}(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$.

Semantics

$\Gamma \models_{\mathbf{AL}^r} A$ iff (for each $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$, if $\text{Ab}(M) \subseteq U(\Gamma)$, then $M \models A$).

Links between the marking and the semantic selection: Minimal Abnormality

Syntax

Theorem 15

$\Gamma \vdash_{\mathbf{AL}^m} A$ iff for every $\varphi \in \Phi(\Gamma)$ there is a $\Delta \subseteq \Omega$ for which $\Delta \cap \varphi = \emptyset$ and $\Gamma \vdash_{\mathbf{LLL}} A \vee \text{Dab}(\Delta)$.

Semantics

Theorem 16

Let $\mathcal{M}_{\mathbf{LLL}}(\Gamma)$ be non-empty.

1. $\mathcal{M}_{\mathbf{AL}^m}(\Gamma) = \bigcup_{\varphi \in \Phi(\Gamma)} \{M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) \mid \text{Ab}(M) = \varphi\}$
2. $\varphi \in \Phi(\Gamma)$ iff there is an $M \in \mathcal{M}_{\mathbf{AL}^m}(\Gamma)$ for which $\text{Ab}(M) = \varphi$.

Conflicts in adaptive proofs

A conflict between a defeasible inference and a “hard fact”

- ▶ “hard facts”: derived on empty condition
- ▶ **Type 1**: hard facts conflict with defeasible assumptions
 - ▶ \rightarrow marking
- ▶ **Type 2**: hard facts conflict with defeasible conclusions

$/$	A	\dots	Δ
$/'$	$\neg A$	\dots	\emptyset

- ▶ in this case $\Gamma \vdash_{\text{LLL}} \text{Dab}(\Delta)$
- ▶ line will be marked
- ▶ shortcut rule

$$\frac{\begin{array}{cc} A & \Delta \\ \neg A & \emptyset \end{array}}{\text{Dab}(\Delta) \quad \emptyset} \quad (\text{RC0})$$

Lemma 17

An **AL**-proof contains a line at which A is derived on the condition Δ iff $\Gamma \vdash_{\text{LLL}} A \vee \text{Dab}(\Delta)$.

A conflict between two defeasible inferences

- ▶ **Type 1:** concerning the defeasible assumption

$$\begin{array}{l} / \quad A \qquad \qquad \qquad \dots \quad \Delta \\ /' \quad \text{Dab}(\Delta) \qquad \qquad \dots \quad \Theta \end{array}$$

- ▶ in this case $\Gamma \vdash_{\text{LLL}} \text{Dab}(\Delta \cup \Theta)$
- ▶ shortcut rule:

$$\frac{\begin{array}{c} A \qquad \qquad \Delta \\ \text{Dab}(\Delta) \qquad \Theta \end{array}}{\text{Dab}(\Delta \cup \Theta) \quad \emptyset} \quad (\text{RD1})$$

- ▶ **Type 2:** concerning defeasible consequences

$$\begin{array}{l} / \quad A \qquad \qquad \qquad \dots \quad \Delta \\ /' \quad \neg A \qquad \qquad \qquad \dots \quad \Theta \end{array}$$

- ▶ in this case $\Gamma \vdash_{\text{LLL}} \text{Dab}(\Delta \cup \Theta)$
- ▶ shortcut rule

$$\frac{\begin{array}{c} A \qquad \qquad \Delta \\ \neg A \qquad \qquad \Theta \end{array}}{\text{Dab}(\Delta \cup \Theta) \quad \emptyset} \quad (\text{RD2})$$

Some trouble with the classical connectives

- ▶ we need *some* classical connectives in order to express Dab-formulas (i.e. the classical disjunction)
- ▶ but what if the LLL has already a classical disjunction?
- ▶ suppose \vee is classical and part of the language of the LLL
- ▶ Let $!A =_{\text{df}} \diamond A \wedge \diamond \neg A$
- ▶ Let $\Gamma = \Gamma_1 \cup \Gamma_2$ where
 - ▶ $\Gamma_1 = \{!A_i \vee !A_j \mid 1 \leq i < j\}$
 - ▶ $\Gamma_2 = \{\bigwedge_{i \leq i < j \leq n} (!A_i \vee !A_j) \supset (A \vee !A_{n-1}) \mid 1 < n\}$
- ▶ Note, $\Phi(\Gamma) = \{\varphi_i \mid i > 0\}$ where $\varphi_i = \Omega \setminus \{!A_i\}$.
- ▶ Moreover, $\Gamma \vdash_{\text{LLL}} A \vee !A_i$
- ▶ Hence, for all $M \in \mathcal{M}_{\text{Tm}}(\Gamma)$, $M \models A$ and whence $\Gamma \models_{\text{Tm}} A$.

1	$\neg A_1 \vee \neg A_2$	PREM	\emptyset
2	$(\neg A_1 \vee \neg A_2) \supset (A \vee \neg A_1)$	PREM	\emptyset
3	$A \vee \neg A_1$	1, 2; RU	\emptyset
¹ 4	A	3; RC	$\{\neg A_1\}$
5	$\neg A_1 \vee \neg A_3$	PREM	\emptyset
6	$\neg A_2 \vee \neg A_3$	PREM	\emptyset
7	$\bigwedge_{1 \leq i < j \leq 3} (\neg A_i \vee \neg A_j) \supset (A \vee \neg A_2)$	PREM	\emptyset
8	$A \vee \neg A_2$	1, 4, 6, 7; RU	\emptyset
⁶ 9	A	8; RC	$\{\neg A_2\}$

- ▶ $\Phi_4(\Gamma) = \{\{\neg A_1\}, \{\neg A_2\}\}$
- ▶ $\Phi_9(\Gamma) = \{\{\neg A_1, \neg A_2\}, \{\neg A_1, \neg A_3\}, \{\neg A_2, \neg A_3\}\}$
- ▶ $\Gamma \not\vdash_{\mathbf{AL}^m}^* A$

How to save the day?

- ▶ classical “checked” symbols are superimposed on the language of LLL
- ▶ where \mathcal{W} is the set of wffs of the LLL, \mathcal{W}^+ is the $\langle \check{\vee}, \check{\wedge}, \check{\neg}, \dots \rangle$ -closure of \mathcal{W}
- ▶ premise sets are considered to be formulated in \mathcal{W}
- ▶ sometimes authors distinguish btw. **LLL** and **LLL⁺**
- ▶ Dab-formulas are formulated with $\check{\vee}$
- ▶ Why does this solve our problem?

1	$\neg A_1 \vee \neg A_2$	PREM	\emptyset
2	$(\neg A_1 \vee \neg A_2) \supset (A \vee \neg A_1)$	PREM	\emptyset
3	$A \vee \neg A_1$	1, 2; RU	\emptyset
4	A	3; RC	$\{\neg A_1\}$

- ▶ line 4 is not marked anymore since $\neg A_1 \vee \neg A_2$ is not a Dab-formula

The Upper Limit Logic

- ▶ Recall: the upper limit logic rigorously interprets the premises normal
- ▶ hence, $\vdash_{\mathbf{ULL}} \neg A$ for all $A \in \Omega$
- ▶ the consequence relation of the upper limit logic is then defined as follows:

$$\Gamma \vdash_{\mathbf{ULL}} A \text{ iff } \Gamma \cup \{\neg A \mid A \in \Omega\} \vdash_{\mathbf{LLL}} A$$

- ▶ semantically **ULL** is characterized by all **LLL**-models M of Γ that are “normal”, i.e. that have an empty abnormal part, $\text{Ab}(M) = \emptyset$.
- ▶ these are precisely the **LLL**-models of $\Gamma \cup \{\neg A \mid A \in \Omega\}$.

ALs approximate ULL

Theorem 18

$$Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{AL}}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(\Gamma)$$

Definition 19

A premise set Γ is *normal* iff it has one of the following equivalent properties

1. $\Gamma \cup \{\neg A \mid A \in \Omega\}$ is **LLL**-non-trivial
2. there are **LLL**-models M of Γ that are normal, i.e. for which $\text{Ab}(M) = \emptyset$

Theorem 20

If Γ is normal, then $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$.

If a premise set can rigorously be interpreted as normal, then the adaptive logic does so.

Properties of the Standard Format

Theorem 21 (Soundness and Completeness)

$\Gamma \vdash_{\mathbf{AL}} A$ iff $\Gamma \models_{\mathbf{AL}} A$.

Theorem 22 (Reflexivity)

$\Gamma \subseteq Cn_{\mathbf{AL}}(\Gamma)$

Theorem 23 (Hierarchy of the Consequence Relations)

$Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{AL}^r}(\Gamma) \subseteq Cn_{\mathbf{AL}^m}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(\Gamma)$

Theorem 24 (Redundancy of **LLL** w.r.t. **AL**)

$Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL}}(\Gamma)) = Cn_{\mathbf{AL}}(\Gamma)$

Theorem 25

$Cn_{\mathbf{AL}}(Cn_{\mathbf{LLL}}(\Gamma)) = Cn_{\mathbf{AL}}(\Gamma)$

Theorem 26 (Fixed Point)

$Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{AL}}(Cn_{\mathbf{AL}}(\Gamma))$

Properties of the Standard Format

Theorem 27 (Cautious Cut / Cumulative Transitivity)

If $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$ then $Cn_{\mathbf{AL}}(\Gamma \cup \Gamma') \subseteq Cn_{\mathbf{AL}}(\Gamma)$.

Theorem 28 (Cautious Monotonicity)

If $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$ then $Cn_{\mathbf{AL}}(\Gamma) \subseteq Cn_{\mathbf{AL}}(\Gamma \cup \Gamma')$.

Corollary 29 (Cautious Indifference)

If $\Gamma \subseteq Cn_{\mathbf{AL}}(\Gamma)$ then $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{AL}}(\Gamma \cup \Gamma')$.

Theorem 30 (Non-Monotonicity/Non-Transitivity)

If $Cn_{\mathbf{LLL}}(\Gamma) \subset Cn_{\mathbf{AL}}(\Gamma)$ then \mathbf{AL} is non-monotonic and non-transitiv.

The “rational” properties

Theorem 31

*In general **AL** is not rational monotonous, i.e. the following does **not** hold:*

If $A \in Cn_{\mathbf{AL}}(\Gamma)$ and $A \notin Cn_{\mathbf{AL}}(\Gamma \cup \{B\})$, then $\neg B \in Cn_{\mathbf{AL}}(\Gamma)$

Theorem 32

*Rational distributivity does not hold for ALs in general, i.e. the following does **not** hold:*

If $A \notin Cn_{\mathbf{AL}}(\Gamma \cup \{B\})$ and $A \notin Cn_{\mathbf{AL}}(\Gamma \cup \{C\})$, then $A \notin Cn_{\mathbf{AL}}(\Gamma \cup \{B \wedge C\})$

Some open questions for you

What about some well-known weakenings of Rational Monotonicity?

- ▶ If $B \in Cn_{\mathbf{L}}(\Gamma)$ and $\neg(B \wedge C) \notin Cn_{\mathbf{L}}(\Gamma)$, then $B \in Cn_{\mathbf{L}}(\Gamma \cup \{C\})$. (proposed by Lou Goble)
- ▶ If $B \in Cn_{\mathbf{L}}(\Gamma)$ and $\neg B \notin Cn_{\mathbf{L}}(\Gamma \cup \{C\})$, then $B \in Cn_{\mathbf{L}}(\Gamma \cup \{C\})$. (proposed by Giordano et al.)



Goble L., A Proposal for Dealing with Deontic Dilemmas, in A. Lomuscio, D. Nute (eds), *DEON*, vol. 3065 of *Lecture Notes in Computer Science*, Springer, p. 74-113, 2004.



Giordano L., Olivetti N., Gliozzi V., Pozzato G. L., ALC + T: a Preferential Extension of Description Logics., *Fundamenta Informatica* p. 341-372, 2009.

Other strategies: the simple strategy

- ▶ applicable in case all minimal Dab-consequences are abnormalities
- ▶ then: $U(\Gamma) = \Phi(\Gamma)$ and hence the reliability strategy and the minimal abnormality strategy result in the same consequence set
- ▶ then: all adaptive models have the same abnormal part
- ▶ simplified marking condition
- ▶ semantic selection ala minimal abnormality or reliability (both select the same models in this case)
- ▶ Task: understand why.

Definition 33 (Marking for the Simple Strategy)

A line l with condition Δ is marked at stage s iff some $B \in \Delta$ is derived on the empty condition.

Definition 34 (Marking for the Simple Strategy 2)

A line l with condition Δ is marked at stage s iff for some $\Delta' \subseteq \Delta$, $\text{Dab}(\Delta')$ is derived on the empty condition.

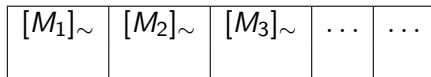
Other strategies: normal selections

- ▶ Rescher-Manor consequence relations:
 - ▶ **strong**: $\bigcap \text{MCS}(\Gamma)$
 - ▶ **weak**: $\bigcup \text{MCS}(\Gamma)$
- ▶ Default reasoning
 - ▶ **skeptical**: in all extensions of the given default theory
 - ▶ **credulous**: in some extension of the given default theory
- ▶ Abstract argumentation
 - ▶ **skeptical**: in all extensions of a given argumentation framework
 - ▶ **credulous**: in some extension of a given argumentation framework
- ▶ Adaptive Logics
 - ▶ **standard format**: $\bigcap_{M \in \mathcal{M}_{\text{AL}}(\Gamma)} \{A \mid M \models A\}$
 - ▶ **normal selections**

Normal Selections Strategy: going “weak” resp. “credulous”

Semantics

- ▶ equivalence relation on the **LLL**-models: $M \sim M'$ iff $\text{Ab}(M) = \text{Ab}(M')$
- ▶ partition of the minimally abnormal models:



- ▶ $\Gamma \Vdash_{\mathbf{AL}}^n A$ iff there is a $M \in \mathcal{M}_{\mathbf{AL}^m}(\Gamma)$ such that for all $M' \in [M]_{\sim}$, $M' \models A$.

Normal Selections

Note: not what is valid in some adaptive model is a consequence!

$\Gamma = \{!A \vee !B, X \vee !A\}$. Minimally abnormal models:

- ▶ models with abnormal part $\{!A\}$:
 - ▶ some validate C (some arbitrary non-abnormal formula)
 - ▶ some validate $\neg C$
- ▶ models with abnormal part $\{!B\}$: these validate X .

We have $\Gamma \Vdash_{\mathbf{AL}}^n X$ but $\Gamma \not\Vdash_{\mathbf{AL}}^n C$.

Normal Selections: Marking

Definition 35 (Marking for Normal Selections)

A line l with condition Δ is marked at stage s iff $\text{Dab}(\Delta)$ is derived on the empty condition at stage s .

Take $\Gamma = \{!A \vee !B, X \vee !A, Y \vee !A \vee !B\}$.

1	$!A \vee !B$	PREM	\emptyset
2	$X \vee !A$	PREM	\emptyset
3	$Y \vee !A \vee !B$	PREM	\emptyset
4	X	2; RC	$\{!A\}$
⁶ 5	Y	3; RC	$\{!A, !B\}$
6	$!A \check{\vee} !B$	1; RC	\emptyset

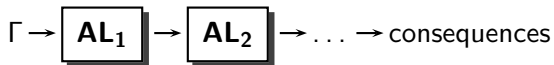
Combining ALs

References:

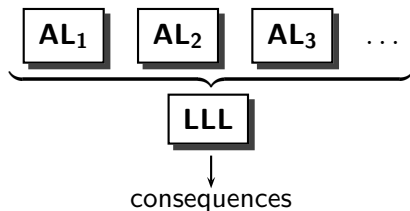
- ▶ Diderik Batens' forthcoming book
- ▶ Frederik Van De Putte, Hierarchic Adaptive Logics [Logic Journal of the IGPL, 2011]
- ▶ Frederik Van De Putte and Christian Straßer, Extending the Standard Format of Adaptive Logics to the Prioritized Case [Logique et Analyse, To appear]
- ▶ Frederik Van De Putte and Christian Straßer, Three Formats of Prioritized Adaptive Logics: a Comparative Study [Under review,]

Combining ALs

1. diachronic combinations / sequential combination / vertical combination / superposing ALs



2. synchronic combinations / horizontal combination / **HAL**



Sequential Combinations

Consequence sets

- ▶ finite case:

$$Cn_{\text{SAL}}(\Gamma) = Cn_{\text{AL}_n^{s_n}} \left(Cn_{\text{AL}_{n-1}^{s_{n-1}}} \left(\dots Cn_{\text{AL}_2^{s_2}} \left(Cn_{\text{AL}_1^{s_1}}(\Gamma) \right) \dots \right) \right)$$

- ▶ infinite case:

$$Cn_{\text{SAL}_i}(\Gamma) = Cn_{\text{AL}_i^{s_i}} \left(\dots \left(Cn_{\text{AL}_2^{s_2}} \left(Cn_{\text{AL}_1^{s_1}}(\Gamma) \right) \right) \right)$$

This is generalized to the infinite case as follows:

$$Cn_{\text{SAL}}(\Gamma) = \liminf_{i \rightarrow \infty} Cn_{\text{SAL}_i}(\Gamma) = \limsup_{i \rightarrow \infty} Cn_{\text{SAL}_i}(\Gamma)$$

Sequential Combinations

Semantics

- ▶ take all **AL**₁-models: \mathcal{M}_1
- ▶ in case $s_2 = m$ take all minimally abnormal models (w.r.t. Ω_2) from \mathcal{M}_1
- ▶ in case $s_2 = r$ take all reliable models (w.r.t Ω_2) from \mathcal{M}_1 :
select all models $M \in \mathcal{M}_1$ for which
 $\text{Ab}(M) \subseteq \bigcup \{ \text{Ab}(M') \mid M' \in \mathcal{M}_2^m \}$ where \mathcal{M}_2^m is the set of all
minimally abnormal models (w.r.t. Ω_2) from \mathcal{M}_1
- ▶ this way we get \mathcal{M}_2
- ▶ repeat this procedure until you're through with all the ALs in
the sequence

Problems with Sequential Combinations: No Fixed Point

Suppose we have $s_1 = s_2 = r$ and

$\Gamma = \{!A_1 \vee !A_2, !A_1 \vee !B, X \vee !A_2\}$ where $!A_1, !A_2 \in \Omega_1 \setminus \Omega_2$ and $!B \in \Omega_2 \setminus \Omega_1$. Take a look at the following **AL**₁-proof:

1	$!A_1 \vee !A_2$	PREM	\emptyset
2	$!A_1 \vee !B$	PREM	\emptyset
3	$!A_1 \check{\vee} !A_2$	1; RU	\emptyset
4	$X \vee !A_2$	PREM	\emptyset
³ 5	X	4; RC	$\{!A_2\}$

In **AL**₂ we can proceed as follows (with the premise set $Cn_{\mathbf{AL}_1}(\Gamma)$):

1	$!A_1 \vee !B$	PREM	\emptyset
2	$!A_1$	1; RC	$\{!B\}$

Hence $!A_1 \in Cn_{\mathbf{AL}_2}(Cn_{\mathbf{AL}_1}(\Gamma))$.

Problems with Sequential Combinations: No Fixed Point

1	$\neg A_1 \vee \neg A_2$	PREM	\emptyset
2	$\neg A_1 \vee \neg B$	PREM	\emptyset
3	$\neg A_1 \checkmark \neg A_2$	1; RU	\emptyset
4	$X \vee \neg A_2$	PREM	\emptyset
³ 5	X	4; RC	$\{\neg A_2\}$

In \mathbf{AL}_2 we can proceed as follows (with the premise set $Cn_{\mathbf{AL}_1}(\Gamma)$):

1	$\neg A_1 \vee \neg B$	PREM	\emptyset
2	$\neg A_1$	1; RC	$\{\neg B\}$

Hence $\neg A_1 \in Cn_{\mathbf{AL}_2}(Cn_{\mathbf{AL}_1}(\Gamma))$.

Let's now apply \mathbf{AL}_1 to the premise set $Cn_{\mathbf{AL}_2}(Cn_{\mathbf{AL}_1}(\Gamma))$:

1	$\neg A_1$	PREM	\emptyset
2	$X \vee \neg A_2$	PREM	\emptyset
3	X	2; RC	$\{\neg A_2\}$

Now, X is a consequence of $Cn_{\mathbf{AL}_1}(Cn_{\mathbf{AL}_2}(Cn_{\mathbf{AL}_1}(\Gamma)))$ and hence also of $Cn_{\mathbf{AL}_2}(Cn_{\mathbf{AL}_1}(Cn_{\mathbf{AL}_2}(Cn_{\mathbf{AL}_1}(\Gamma))))$.

Problems with Sequential Combinations

- ▶ Note: lack of deduction theorem is the culprit:
 $\Gamma \cup \{!A_1\} \vdash_{\mathbf{AL}_1} X$ but $\Gamma \not\vdash_{\mathbf{AL}_1} B \dot{\supset} X$.
- ▶ This also shows that we don't have Cautious Transitivity:
 $Cn_{\mathbf{SAL}}(\Gamma \cup \{!A\}) \not\subseteq Cn_{\mathbf{SAL}}(\Gamma)$ although $!A \in Cn_{\mathbf{SAL}}(\Gamma)$.

Problems with Sequential ALs: Lack of completeness for minimal abnormality

Let $\Gamma = \{X \vee !A_i \vee !B_i \mid i \in \mathbb{N}\} \cup \{!A_i \vee !A_j \mid i \neq j\}$ and $A_i \in \Omega_1 \setminus \Omega_2$ and $B_i \in \Omega_2 \setminus \Omega_1$. Take a look at the following \mathbf{AL}_1^m -proof from Γ :

1	$X \vee !A_1 \vee !B_1$	PREM	\emptyset
⁴ 2	$X \vee !B_1$	1; RC	$\{!A_1\}$
3	$!A_1 \vee !A_2$	PREM	\emptyset
4	$!A_1 \checkmark !A_2$	3; RU	\emptyset

Hence, $X \vee !B_i$ is not derivable for any $i \in \mathbb{N}$.

Take a look at the following \mathbf{AL}_2^m -proof from $Cn_{\mathbf{AL}_1^m}(\Gamma)$:

1	$X \vee !A_1 \vee !B_1$	PREM	\emptyset
2	$X \vee !A_1$	1; RC	$\{!B_1\}$

Hence, $X \vee !A_1 \in Cn_{\mathbf{AL}_2^m}(Cn_{\mathbf{AL}_1^m}(\Gamma))$ but there is no way of deriving X .

Problems with Sequential ALs: Lack of completeness for minimal abnormality

- ▶ Now let's take a look at the semantic selection.
- ▶ $\mathcal{M}_{\mathbf{AL}_1^m}(\Gamma) = \{M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) \mid \text{Ab}_1(M) = \{!A_i\}, i \in \mathbb{N}\}$.
- ▶ Hence, for each $M \in \mathcal{M}_{\mathbf{AL}_1^m}(\Gamma)$, $M \models X \vee !B_i$ for some $i \in \mathbb{N}$.
- ▶ $\mathcal{M}_2 = \{M \in \mathcal{M}_{\mathbf{AL}_1^m}(\Gamma) \mid \text{Ab}_2(M) = \emptyset\}$. Hence, for all $M \in \mathcal{M}_2$, $M \models X$.
- ▶ Hence $X \notin \text{Cn}_{\mathbf{SAL}}(\Gamma)$ but $\Gamma \Vdash_{\mathbf{SAL}} X$.

Restricted positive results

- ▶ Suppose $\Omega_1 \subseteq \Omega_2 \subseteq \dots$. Then, **SAL** is sound.
- ▶ If one of the following holds, then **SAL** is sound and complete, has a fixed point, is cautious transitive, etc. See REF. Let $\Sigma(\Gamma)$ and $\Phi(\Gamma)$ be the corresponding sets of the AL $\langle \mathbf{LLL}, \bigcup_{i \in \mathbb{N}} \Omega_i, m \rangle$.
 1. $\Sigma(\Gamma)$ is finite
 2. every $\varphi \in \Phi(\Gamma)$ is finite
 3. $\Phi(\Gamma)$ is finite

Prioritized ALs

Prioritized abnormalities

- ▶ Ω_1 : contains the ones we want to avoid mostly
- ▶ Ω_i : having a choice between a level i abnormality and higher order level abnormality we'd choose the level i , but would prefer higher level abnormalities

Prioritized Format for ALs

- ▶ Lower limit logic: **LLL**
- ▶ sequence of abnormalities: $\langle \Omega_i \rangle_{i \in I}$
- ▶ adaptive strategy: minimal abnormality and reliability

Prioritized ALs

Lexicographic order

- ▶ used e.g. in telephone book: compare sequences component-wise, as soon as one scores better it's preferred
- ▶ general: Suppose we have a sequence of linear orders: $\langle (X_i, <_i) \rangle_I$ (where $I \subseteq \mathbb{N}$). Then define $<_{\text{lex}} \subseteq \times_I X_i$ as follows: $\langle a_i \rangle_I <_{\text{lex}} \langle b_i \rangle_I$ iff there is a minimal j such that (i) $a_k = b_k$ for all $k < j$ and (ii) $a_j <_j b_j$.
- ▶ compare prioritized sequences of sets of abnormalities:
Let $\langle \Delta_i \rangle_I, \langle \Delta'_i \rangle_I \in \times_I \wp(\Omega_i)$.
 $\langle \Delta_i \rangle_I \sqsubset_{\text{lex}} \langle \Delta'_i \rangle_I$ iff (i) there is an $i \in I$ such that for all $j < i$, $\Delta_j = \Delta'_j$ and (ii) $\Delta_i \subset \Delta'_i$.
Let $\Delta, \Delta' \in \wp(\bigcup_I \Omega_i)$.
We write $\Delta \sqsubset \Delta'$ for $\langle \Delta \cap \Omega_i \rangle_I \sqsubset_{\text{lex}} \langle \Delta' \cap \Omega_i \rangle_I$.

Prioritized ALs: Semantics

Definition 36

$\text{Ab}(M) = \{A \in \Omega \mid M \models A\}$ where $\Omega = \bigcup_i \Omega_i$.

$M \in \mathcal{M}_{\mathbf{AL}_{\sqsubseteq}^m}(\Gamma)$ iff $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ and there is no $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ such that $\text{Ab}(M') \sqsubset \text{Ab}(M)$.

Alternative Characterization

- ▶ let $\Phi^{\sqsubseteq}(\Gamma)$ be the \sqsubseteq -minimal choice sets of $\Sigma(\Gamma)$.
- ▶ note: $\Delta \sqsubset \Delta'$ then $\Delta \subset \Delta'$ and $\Phi^{\sqsubseteq}(\Gamma) \subseteq \Phi(\Gamma)$
- ▶ $M \in \mathcal{M}_{\mathbf{AL}_{\sqsubseteq}^m}(\Gamma)$ iff $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ and $\text{Ab}(M) \in \Phi^{\sqsubseteq}(\Gamma)$.

Other alternative semantic characterizations

Suppose in the following that $\Omega_i \subseteq \Omega_{i+1}$ for all $i, i + 1 \in I$. Then we have two more alternative semantic characterizations:

1. sequential selections:

- ▶ $M[0]$ = set of LLL-models of Γ
- ▶ for each i in I do
 - ▶ $M[i]$ = the set of all minimal abnormal models in $M[i - 1]$ w.r.t. Ω_i

2. intersecting: $\mathcal{M}_{\mathbf{AL}_{\sqsubseteq}^m}(\Gamma) = \bigcap_I \mathcal{M}_{\mathbf{AL}_i^m}(\Gamma)$.

Prioritized ALs: Proof theory

- ▶ same generic rules as proofs in standard format
- ▶ let $\Phi_s^\square(\Gamma)$ be the set of \square -minimal choice sets of $\Sigma_s(\Gamma)$

Definition 37

A line l with formula A and condition Δ is marked iff (i) no $\varphi \in \Phi_s^\square(\Gamma)$ is such that $\varphi \cap \Delta = \emptyset$, or (ii) for a $\varphi \in \Phi_s^\square(\Gamma)$ there is no line on which A is derived on a condition Θ for which $\Theta \cap \varphi = \emptyset$.

Let $\Gamma = \{X \vee !A_i \vee !B_i \mid i \in \mathbb{N}\} \cup \{!A_i \vee !A_j \mid i \neq j\}$ and $A_i \in \Omega_1 \setminus \Omega_2$ and $B_i \in \Omega_2 \setminus \Omega_1$.

1	$X \vee !A_1 \vee !B_1$	PREM	\emptyset
2	X	1; RC	$\{!A_1, !B_1\}$
3	$!A_1 \vee !A_2$	PREM	\emptyset
4	$X \vee !A_2 \vee !B_2$	PREM	\emptyset
5	X	4; RC	$\{!A_2, !B_2\}$

Note that $\Phi^{\square}(\Gamma) = \{\Omega_1 \setminus \{!A_i\} \mid i \in \mathbb{N}\}$. Since we can derive X on the condition $\{!A_i, !B_i\}$ for each $i \in \mathbb{N}$, $\Gamma \vdash_{\mathbf{AL}_E^m} X$.

$\Gamma = \{!A \vee !B, !A \vee !C, X \vee !A, Y \vee !B\}$. Let $!A \in \Omega_1$, $!B \in \Omega_2 \setminus \Omega_1$ and $!C \in \Omega_3 \setminus (\Omega_1 \cup \Omega_2)$.

1	$!A \vee !B$	PREM	\emptyset
2	$!A \vee !C$	PREM	\emptyset
3	$X \vee !A$	PREM	\emptyset
4	$Y \vee !B$	PREM	\emptyset
5	X	3; RC	$\{!A\}$
⁷ 6	Y	4; RC	$\{!B\}$
7	$!A \check{\vee} !B$	1; RU	\emptyset
8	$!A \check{\vee} !C$	2; RU	\emptyset

We have $\Phi_8^{\square}(\Gamma) = \{\{!B, !C\}\}$ since $\{!B, !C\} \sqsubset \{!A\}$.

Prioritized ALs: Meta-Theory

- ▶ very rich: similar to standard format, e.g.
- ▶ soundness and completeness
- ▶ Strong reassurance
- ▶ reflexivity
- ▶ cautious indifference
- ▶ fixed point
- ▶ if Γ is normal, then $Cn_{\mathbf{AL}_{\perp}^m}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$.