Recall: What was the problem with $A \supset OB$ and $O(A \supset B)$?

$\begin{array}{|c|c|}
\hline
A \supset OB & O(A \supset B) \\
\hline
1 & \text{problem with exceptions} & \text{problem with exceptions} \\
2 & \text{logical dependency} & \text{logical dependency} \\
3 & \text{contraposition} & \text{contraposition} \\
4 & \text{no transitivity-(like) prop.} & \text{no factual detachment} \\
\hline
\end{array}$

- $A \supset OB$ implies $(A \land X) \supset OB$
- $\neg A$ implies $A \supset OB$
- $\neg OB \supset \neg A$ (see Caminada’s criticism)
- from $A$ and $O(A \supset B)$ doesn’t follow $OB$

Using preferential semantics for a dyadic version of SDL

- $M = \langle W, \leq, v \rangle$ where $\leq$ is a connected preorder (reflexive and transitive)
- $M, w \models O(B/A)$ iff there is a $w' \in F \leq w$ such that $M, w' \models A \land B$
- $M, w \models A \land B$ and for any $w''$ for which $w' \leq w''$: $M, w'' \models A$ implies $M, w'' \models B$.
- axiomatized by:
  - RCE If $\vdash A \equiv A'$ then $\vdash O(B/A) \equiv O(B/A')$
  - RCM If $\vdash B \supset C$ then $\vdash O(B/A) \supset O(C/A)$
  - CK $O(B \supset C/A) \supset (O(B/A) \supset O(C/A))$
  - CD $O(B/A) \supset \neg O(\neg B/A)$
  - CN $O(\top/\top)$
  - $CO \land O(B/A) \supset O(A \land B/A)$
  - trans $((A \geq B) \land (B \geq C)) \supset (A \geq C)$ where $A \geq B = df \neg O(\neg A/A \lor B)$ (read: “$A$ is at least as good as $B$”)
Checking properties...

Check whether the following principles hold:

\[
\begin{align*}
\text{AND} & \quad \frac{O(A \mid C)}{O(A \land B \mid C)} \\
\text{OR} & \quad \frac{O(A \mid B)}{O(A \mid B \lor C)} \\
\text{WC (Weakening of the Consequent)} & \quad \frac{O(A \mid B \land C)}{O(A \mid B)} \\
\text{ID (tautological ID)} & \quad O(A \mid A) \\
\text{IDT (tautological ID)} & \quad O(\top \mid \top)
\end{align*}
\]

Do we get some problems?

- Can we model CTD-cases?
- Can you think of an example where RM is too strong?
- Think about the problems we had for the unary approaches \((A \supset OB)\) and \(O(A \supset B)\) – do we still have any problems like that?

Default theory

\[\langle F, \Delta \rangle\]

- \(F\): Facts
- \(\Delta\): set of conditional obligations, written as \(A \rightarrow B\) (note: this is not the material implication!)
Extensions

- Idea: iterative procedure
- apply factual and deontic detachment step-by-step
- here's how it works:

The procedure

- step 0
  - let \( E_0 = F \)
  - pick a \( A \rightarrow B \) from \( \Delta \) that is (1) triggered by \( E_0 \) and (2) not conflicted by \( E_0 \)
    - \( A \rightarrow B \) is triggered by \( E \) iff \( E \vdash A \)
    - \( A \rightarrow B \) is conflicted by \( E \) iff \( E \vdash \neg B \)
  - let \( E_1 = E_0 \cup \{ B \} \)

- step \( i + 1 \)
  - pick a \( A \rightarrow B \) from \( \Delta \) that is (1) triggered by \( E_i \) and (2) not conflicted with \( E_i \)
  - let \( E_{i+1} = E_i \cup \{ B \} \)

Do this until a fixed-point is reached (that is: nothn new is added anymore).

Let \( \langle F, \Delta \rangle \) where

- \( F = \{ a \} \)
- \( \Delta = \{ a \rightarrow b, b \rightarrow c, a \rightarrow \neg c \} \)

What are the extensions of this theory?

How to define a consequence relation?

- credulous approach
- skeptical approach

Examples

Let \( \langle F, \Delta \rangle \) where

- \( F = \{ n \} \)
- \( \Delta = \{ n \rightarrow q, n \rightarrow r, q \rightarrow p, r \rightarrow \neg p \} \)

What can we derive from that?
Yet another example

Let \( \langle F, \Delta \rangle \) where

- \( F = \{a \lor b\} \)
- \( \Delta = \{a \rightarrow c, b \rightarrow c\} \)
- What can we derive?

Prioritized default theories

\( \langle F, \Delta, \prec \rangle \)

where we have a strict partial order on the defaults: \( \prec \).

- This represents a priority/preference relation.
- Depending on the application this may indicate the rank of an authority from which the information stems, the reliability of the source, specificity relations, etc.

A strict partial order is (i) irreflexive (\( A \not< A \)), (ii) asymmetric (If \( A < B \) then \( B \not< A \)), and (iii) transitive (\( A < B \) and \( B < C \) implies \( A < C \)). Graphically they are represented by directed acyclic graphs (the transitive closure is usually not represented).

Scenarios vs. extensions

Given an ordered default theory \( \langle \Phi, \Delta, \prec \rangle \), Horty distinguishes between

- scenarios, that is sets of defaults in \( \Delta \), and
- the sets of beliefs generated by scenarios given the facts \( \Phi \), i.e., \( \text{Cn}(\Phi \cup \text{Conclusion}(S)) \) where \( S \) is a scenario.

- like in the non-prioritised case, not just any superset of \( \Phi \) constitutes an extension, we are also now interested in scenarios that in some sense represent the set of defaults a rational agent would select/use given \( \Phi \) and \( \prec \).
- E.g., a scenario should not generate conflicting beliefs.
- Moreover, priorities should be taken into account.
- Hence, we are interested in what Horty calls proper scenarios and the belief sets generated by them.
The idea is again to build up scenarios stepwise similar as in the procedural approaches to build extensions.

we start with our facts $\Phi$

What are interesting defaults to take into account?

As in Reiter’s default logic we have the choice between triggered defaults:

$$\text{Triggered}(S) = \{ \delta \in \Delta \mid \Phi \cup \text{Conclusion}(S) \vdash \text{Premise}(\delta) \}$$

However, some triggered defaults may bad candidates since they (i.e., their conclusions) conflict with our belief set. Hence, we want to neglect:

$$\text{Conflicted}(S) = \{ \delta \in \Delta \mid \Phi \cup \text{Conclusion}(S) \vdash \neg \text{Conclusion}(\delta) \}$$

Also, we have to take into account our priorities.

A first idea would be to pick one of the highest ranked triggered defaults.

this is not Horty’s approach (see later: the order puzzle)

We also have to take into account that sometimes sets of defaults “defeat” other defaults

$$S < S'$$

where $S, S' \subseteq \Delta$ and $\delta < \delta'$ for all $\delta \in S$ and all $\delta' \in S'$

E.g., where $\delta_1 = a \rightarrow b$, $\delta_2 = b \rightarrow c$ and $\delta_3 = a \rightarrow \neg c$ and $\delta_3 < \delta_1, \delta_2$,

we have $\{\delta_3\} < \{\delta_1, \delta_2\}$

Note that $\{a\} \cup \text{Conclusion}((\delta_1, \delta_2)) \vdash \neg \text{Conclusion}(\delta_3)$

in this sense $\{\delta_1, \delta_2\}$ defeats $\delta_3$

Horty makes the idea above precise relative to a given scenario $S$.

The set Defeated($S$) is the set of all $\delta \in \Delta$ such that there is a $\Delta_d \subseteq \text{Triggered}(S)$ (a defeating set) for which

1. $\{\delta\} < \Delta_d$
2. there is a $S_a \subseteq S$ (an accommodation set) for which
   1. $S_a < \Delta_d$
   2. $\Phi \cup \text{Conclusion}((S \setminus S_a) \cup \Delta_d)$ is consistent
   3. $\Phi \cup \text{Conclusion}((S \setminus S_a) \cup \Delta_d) \vdash \neg \text{Conclusion}(\delta)$

Some examples

Take $T = (\{P, P \supset B\}, \{\delta_1, \delta_2\}, \{(\delta_1, \delta_2)\})$ where $\delta_1 = B \rightarrow F$ and $\delta_2 = P \rightarrow \neg F$.

Suppose $S = \{B \rightarrow F\}$.

- $P \rightarrow \neg F \in \text{Triggered}(S)$
- $\{B \rightarrow F\}$ is an accommodation set for the defeating set $\{P \rightarrow \neg F\}$ w.r.t. $B \rightarrow F$
Given an ordered default theory \( \langle \Phi, \Delta, < \rangle \) we construct proper scenarios as follows.

- **Guess** \( S \)

- the initial scenario is \( S_0 = \emptyset \)

- do the following until a fixed point is reached
  - \( S_{i+1} \): add all \( \delta \in \Delta \) to \( S_i \) that satisfy the following conditions
    - \( \delta \in \text{Triggered}(S) \)
    - \( \delta \notin \text{Conflicted}(S) \) (here you need to make use of your guess!)
    - \( \delta \notin \text{Defeated}(S) \) (also here)

- let your fixed point be \( S' \).

  - If \( S = S' \) your done. Then \( S \) is a proper scenario and
    - \( \Xi = \text{Cn}(\Phi \cup \text{Conclusion}(S)) \) is an extension.
  - Otherwise, start anew with another guess.

An example

Take \( T = \langle \{ P, P \supset B \}, \{ \delta_1, \delta_2 \}, \{ (\delta_1, \delta_2) \} \rangle \) where \( \delta_1 = B \supset F \) and \( \delta_2 = P \supset \neg F \).

- **Guess**: \( S = \{ B \supset F \} \).
  - First round:
    - \( S_0 = \emptyset \)
    - \( B \supset F, P \supset \neg F \in \text{Triggered}(S_0) \)
    - \( P \supset \neg F \in \text{Conflicted}(S) \)
    - but: \( B \supset F \notin \text{Defeated}(S) \)
    - nothing is added!

- **Guess**: \( S = \{ P \supset \neg F \} \).
  - First round:
    - \( S_0 = \emptyset \)
    - \( B \supset F, P \supset \neg F \in \text{Triggered}(S_0) \)
    - \( B \supset F \in \text{Conflicted}(S) \)
    - \( P \supset \neg F \notin \text{Defeated}(S) \)
    - hence: \( S_1 = \{ P \supset \neg F \} \)
  - second round: fixed point reached.

An intuitive problem?

Take \( T = \langle \Phi, \Delta, < \rangle \) where

- \( \Delta = \{ \delta_1, \delta_2, \delta_3 \} \) with
  - \( \delta_1 = a \supset b \)
  - \( \delta_2 = b \supset c \)
  - \( \delta_3 = c \supset \neg b \)

- \( \Phi = \{ a \} \)
- \( \delta_3 < \delta_1 \) and \( \delta_3 < \delta_2 \)
- Let \( S_1 = \{ a \supset \neg c \} \)
  - \( a \supset b \) is triggered by \( S \)
  - no defeating takes place
- Let \( S_2 = \{ a \supset \neg c, a \supset b \} \)
  - \( b \supset c \) is triggered
  - \( b \supset c \) is conflicted by \( S \)
  - \( \{ a \supset \neg c \} \) is defeated by \( S \)
  - note that \( \{ a \supset \neg c \} \) is an accommodation set for the defeating set \( \{ b \supset c \} \)
Take $T = \langle \Phi, \Delta, < \rangle$ where
- $\Delta = \{ \delta_1, \delta_2, \delta_3 \}$ and
  - $\delta_1 = a \rightarrow b$
  - $\delta_2 = b \rightarrow c$
  - $\delta_3 = a \rightarrow \neg c$
- $\Phi = \{ a \}$
- $\delta_3 < \delta_1$ and $\delta_3 < \delta_2$
- Guess: $S = \{ a \rightarrow \neg c \}$
  - Round 1: $S_0 = \emptyset$
  - Triggered: $a \rightarrow b$ and $a \rightarrow \neg c$
  - $a \rightarrow b \notin \text{Defeated}(S)$
  - $a \rightarrow b \notin \text{Conflicted}(S)$
  - hence, $a \rightarrow b$ has to be added!
- Guess: $S = \{ a \rightarrow b, b \rightarrow c \}$
  - This works: check out why

**Theorem**
Where $\langle \Phi, \Delta, < \rangle$ is an ordered default theory and $< = \emptyset$, then the associated extensions are exactly the same as the Reiter-extensions of $\langle \Phi, \Delta' \rangle$ (where $\Delta'$ is the translation of $\Delta$ into normal defaults).

**Theorem**
Where $T_\prec = \langle \Phi, \Delta, \prec \rangle$ is an ordered default theory and $\Xi$ is an extension of $T_\prec$, then $\Xi$ is also a Reiter-extension of $\langle \Phi, \Delta' \rangle$ (where $\Delta'$ is the translation of $\Delta$ into normal defaults).