For Sergio Alberverio on occasion of his 60th birthday!

Space, Interaction, and Gauge Invariance

Michael Drieschner

ABSTRACT. The claim is analyzed that gauge invariance introduces in some miraculous way interaction into a so far free theory. The result of the analysis is that interaction is put into the theory by hand, although in a somehow hidden way. This is made clearer by a work of PAUL TELLER. The essential role of space for interaction is emphasized, ending with a critical glimpse on the usual interpretation of the AHARONOV-BOHM effect.

1. Introduction

The gauge principle is a powerful tool in modern field theory. It seems that we can introduce an interaction theory in starting from a free field theory and postulating its gauge invariance: By some magic then an interacting field theory comes out.

WOLFGANG PAULI introduced this powerful tool into physics in his famous paper [**8**]. But at that time he did not bother much with logical or structural clarification of his arguments. He begins with a general Lagrangian without an external field, and then rather suddenly introduces interaction via the covariant derivative with the words: "With this assumption it is possible to introduce an external electromagnetic field by replacing the operation $\partial/\partial x$ [...] by the operator $D_k = (\partial/\partial x) - i\epsilon\phi_k$"(p. 206). On the next page he introduces local phase transformations: "The theory obtained in that manner is invariant with respect to the gauge transformation ..., where now α may be an arbitrary function of position." – and that is all he says about 'derivation'.

SUNNY AUYANG in her book [**2**] describes the procedure in detail – we shall describe a similar derivation in the next paragraph – with justifications like: "It is unreasonable to expect that phase changes are always global." [**2**, p. 57]. She continues under the heading of *The Logic of Gauge Field Theories*: "... we start with a free matter field and derive the interacting field system in the following steps ..." [**2**, p. 58]. – This is a rather strong claim, since she talks about *deriving* the interacting field system, and she ought to know what she is talking about, being a philosopher and not a physicist. We shall have a look at her arguments shortly.

But the strongest claim is by Holger Lyre in his paper for the GTR conference of 1999 at Notre Dame[**7**]. He maintains explicitly that "Miraculously, it turns out that this coupling can in fact be *derived* just by postulating the invariance of [the Lagrangian] under local gauge transformations instead of the corresponding global ones." (His italics!)

The purpose of this paper is to have a closer look at such claims and their justification.

I thank Holger Lyre and Tim Eynck for interesting discussions.

2. The Introduction of Gauge Invariance

Let us consider the free DIRAC equation

$$(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0, \tag{2.1}$$

where $\psi(x)$ is a quantum mechanical state function. This equation remains valid if we transform $\psi(x)$ by a phase factor $e^{iq\alpha}$, with constants q, α:

$$\psi(x) \to \psi'(x) = e^{iq\alpha}\psi(x), \qquad \bar{\psi}(x) \to \bar{\psi}'(x) = e^{-iq\alpha}\bar{\psi}(x). \tag{2.2}$$

This is good old quantum mechanics, where all functions that differ only by a phase factor represent the same quantum mechanical state; mathematically a state is represented by a *subspace* of HILBERT space. [**4**][**5**]

Now comes the miraculous trick that is supposed to bring interaction into the game: Instead of a "global" constant α we introduce a "local", i.e. spacetime dependent, phase *function* $\alpha(x)$. The justification of this step usually is rather foggy, saying something like: "Since all physics is local, the phase invariance must be local", or, as SUNNY AUYANG puts it, after the phrase quoted above: "*The symmetry U(1) is localized* to each point x in the field". [**2**, p. 57, her italics]

Taken literally this is all nonsense: Expectation values of all observables are invariant under global phase transformations, as one can easily see, since global phase factors commute with all operators. This is not true any more for a "local" phase factor. Take, for a simple example, [1] $\alpha(\mathbf{x}) = p_0 \cdot x$. This changes, as one can easily calculate, the expectation value of $\hat{p}_x = -i\partial_x$ from p_x to $p_x + p_0$. – To put it more generally: A local phase transformation does, in general, change the expectation value of an operator if that operator contains a differentiation.

Thus let us do what physicists usually do: Try local phase transformations and see what comes out. But – I want to emphasize – be aware of the fact that this transformation *changes* the quantum mechanical state!

Now let us continue the "derivation": The local phase transformation

$$\psi(x) \to \psi'(x) = e^{iq\alpha(x)}\psi(x) \tag{2.3}$$

changes the free DIRAC equation (2.1) into

$$(i\gamma^\mu \partial_\mu - m)\,\psi'(x) = -q \cdot \gamma^\mu \partial_\mu \alpha(x) \cdot \psi'(x). \tag{2.4}$$

We are taking this equation as our new field equation.

Then the physicist's recipe prescribes the following:

Define

$$A_\mu(x) = -\partial_\mu \alpha(x). \tag{2.5}$$

Insert (2.5) into (2.4); this gives (omitting the primes) the equation:

$$(i\gamma^\mu \partial_\mu - m)\,\psi(x) = q \cdot \gamma^\mu A_\mu(x) \cdot \psi(x). \tag{2.6}$$

Now define for $A_\mu(x)$ what WOLFGANG PAULI calls a gauge transformation of the second type[**8**, p. 207]:

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu \alpha(x), \tag{2.7}$$

using the same function $\alpha(x)$ as in (2.3).

If one applies both transformations, (2.3) and (2.7), at once, equation (2.6) is left invariant.

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[1] Only in this paragraph the designation x for the four-dimensional coordinates is changed to $\mathbf{x}$, and x designates one of the space coordinates.

This can be achieved as well by replacing the derivative ∂_μ by the "covariant derivative" $D_\mu = \partial_\mu + iqA_\mu$—as was quoted form PAULI above. It can be shown that $A_\mu(x)$ has all transformation properties of the electromagnetic potential. – This is all you have to do, and, *hey presto*, there is a true enough interaction equation.[2]

3. Magic Uncovered

Let us now have a closer look at the fingers of those magicians: What do they really put into the trick, in order to get interaction out of it?

We start with the free DIRAC equation (2.1): No interaction! Then we put local phase transformations into the trick: First ingredient! The result is (2.6). Now look closely: The right side of (2.5) is a (space-time) gradient. If we interpret it as an electromagnetic potential, it is a case of zero electromagnetic field: So there is still no interaction! Thus, local phase transformation does *not* introduce interaction.

Where does interaction come in, since it is apparently *not* by local phase transformations? – In reality it is a second ingredient, put in by hand, and camouflaged by the innocent looking definition (2.5): Local phase transformation by itself introduces only field zero potentials. But the magician uses that opportunity to generalize the equation and admit *any* electromagnetic potential. This is an extra step in the argument, which is usually not mentioned; but it is by no means self evident. Going on from field zero potentials to really interacting ones is in fact a decision; and this is where interaction comes in!

4. Why Does It Work?

The two types of so-called gauge transformations involved [**8**, p. 207] are of quite different quality: Gauge transformations of the second type are well known from classical electrodynamics; they display the "gauge freedom" of the electromagnetic potential, because different potentials entail the same electric and magnetic fields if their difference is a (four dimensional) gradient. Thus a second type gauge transformation changes only the description, not its physical content; it is conventional. The same is true for the "global" phase factor, but it is not true for the local phase transformations, the "gauge transformations of the first type": They do change the quantum mechanical state, i.e. the physical meaning of the description.

5. The analogy with General Relativity

Paul Teller[3] gives an account of the "gauge argument" that is much more to the point and a lot clearer than the usual "derivation" described above. In his argument he maintains that "gauge transformations of the first type", contrary to their face value, are conventional as well.

Teller uses an analogy with coordinate transformations in geometry: If we have a flat space with Cartesian coordinates, we can calculate the difference between vectors at distant points by subtracting directly their coordinate values. But imagine we had introduced curvilinear coordinates into that (still flat) space. Then, in order to calculate the difference between vectors at distant points we have to introduce a prescription of parallel transport—something that compensates for the consequence of the merely conventional coordinate change. Teller considers "local

[2]For a very reasonable account of the logical structure in the framework of fiber bundles cf. [**9**].

[3][**10**]; Teller refers to MORIASU[**8**].

coordinate transformations" as conventions, analogous to curvilinear coordinates in a flat space: We can introduce local phase transformations if we compensate for their influence in the description of reality—namely, in our case, by adding the A_μ-term to form the covariant derivative.

This step is the same as the first step in the "derivation" described above. It is considered by Teller—convincingly—as merely conventional. After this step equation (2.5) still applies: There are no electromagnetic fields— in analogy with a flat space.

But now the stage is set for the second step, namely for introducing interaction, in the same way as curvilinear coordinates in a flat space set the stage for introducing a non-flat space. The second step is here, replacing the field free A_μ by a really interacting A_μ, with an electromagnetic field $\neq 0$. This is really a new step, as described above, that is not necessitated by the first step. But introducing local phase transformations at first conventionally (like curvilinear coordinates in flat space) makes visible a possibility that was hidden before. In this respect the work of Paul Teller gives really new insight into the secret inner workings of gauge field theory.

6. The role of space in interaction

What does it mean that the interaction equation is invariant under gauge transformations? – The quantum mechanical state is changed physically by the gauge transformation, whereas everything else is even on the surface transformed only conventionally, and remains unchanged as far as physical content is concerned. This means, quantum mechanical states that can be transformed into each other by gauge transformations are valid solutions for the *same* physical situation, especially for the same interacting electromagnetic field.

This is very interesting: What all the ψ functions that are transformed into each other by gauge transformations have in common is their spatial probability distribution. Thus the spatial probability distribution seems to be the essential property for interaction – in this case of the Dirac particle with the electromagnetic field, but since gauge invariance seems to be rather universal, this might be a property of all kinds of interactions.

This is a feature that has been discussed for a long time. C.F. von Weizsäcker claims that W. Heisenberg made a point of it, and at least Weizsäcker himself has discussed it in several places (cf. [**11**, p. 203], [**12**, p. 381]).

This seems to bring us closer to the real explanation of the role of gauge invariance for theories of interaction: Gauge invariance groups quantum mechanical states into equivalence classes all members of which have the same spatial probability distribution, and all states of one equivalence class interact in the same way with the field.

Thus a gauge invariant equation shows most clearly how and where interaction comes in. This might be the true reason behind the introduction of "local" phase transformations.

7. Aharonov-Bohm effect

A topic that is frequently discussed in this environment is the Aharonov-Bohm effect[**1**]. Many people maintain that it shows the direct physical significance of the electromagnetic potential, even of different effects of potentials with the same electromagnetic field; and that this supports the claim of the “miraculous” origin of the interaction from local phase transformations.

But, as we have seen, nothing is miraculous about the introduction of interaction. Gauge invariance just indicates a good place where to put interaction in by hand. And by the same token nothing is miraculous about the influence of the electromagnetic potential in the Aharonov-Bohm effect either; it is really the electromagnetic *field* way outside in space that is responsible for the effect. The reason for the surprise is a double misconception: first one forgets that the field free space is an idealization; but the field integrated over all space outside the solenoid exactly compensates the field within. The second misconception is the classical image of an electron spatially concentrated at its path; but the wave function of the electron is spread out all over space, and its interaction with the electromagnetic field extends all over space as well.

8. Conclusion

Thus, if we look carefully at our concepts of space and interaction, all miracle vanishes out of gauge theory, and out of Aharonov-Bohm as well.

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Institut für Philosophie, Ruhr-Universität Bochum, D-44780 Bochum, Germany
Email address: Michael.Drieschner@ruhr-uni-Bochum.de