A report on how to prove that all planar sets are aspherical

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Abstract: The proof that to a certain extend is described in this paper, is essentially subdivided into two steps: Firstly it is proven that any compact and totally and locally pathwise connected subset of the plane can be homotopy-equivalently embedded into a "Modified Sierpinski Carpet", i.e. a space which can be obtained by sometimes omitting to take out a square in the scheme of constructing the classical Sierpinski carpet. Such a space offers the opportunity of defining infinite combinatorial objects that surject on the variety of all paths. A combinatorial analysis of these objects is then the second main step of the proof. On a combinatorial basis minimal elements in these equivalence classes are be selected, and by using these minimal representatives a system of paths can be selected that connects each point of the plane in each relative homotopy class to the base point. Such a system of paths provides a natural method of contracting a continuous image of a higher dimensional sphere. Thanks to the combinatorial control of our system of paths and to the properties of minimal representatives it can be shown that the above construction actually gives a continuous contraction process. This proves nullhomotopy in an arbitrary situation, hence asphericity.

This is an abridged version of a piece of work where the fully expanded version ([Zas0]) that recently has been accepted as a habilitation theses at the Ruhr-University Bochum (Germany) takes 263 pages. This version outlines the proof and gives all essential ideas and proof-arguments. On the other hand, the technical definitions are only described, proofs are only outlined; and wherever there are special cases, extra discussion may be omitted.

The assertion as stated in the title can be rephrased as follows (cf. 2.21):

> **"Theorem A:** *Let Y be a subset of $\mathbb{R}^2$ with the induced topology, and let $y_0 \in Y$. Then $\pi_k(Y, y_0) = 0$ for $k \geq 2$."*

Such a statement can easily be proved under suitable tameness assumptions on Y. Hence the main bulk of this paper is spent on finding a way of contracting arbitrary continuous images of higher dimensional spheres when tameness assumptions do not hold. More precisely, it is pointed out to what extent some minor tameness conditions can be assumed without loss of generality; and the remaining phenomena of pathology are then discussed and described by a certain combinatorial concept.

There are good reasons to believe that this has been a fairly old problem. However, I have not been able to find references that could establish its age.

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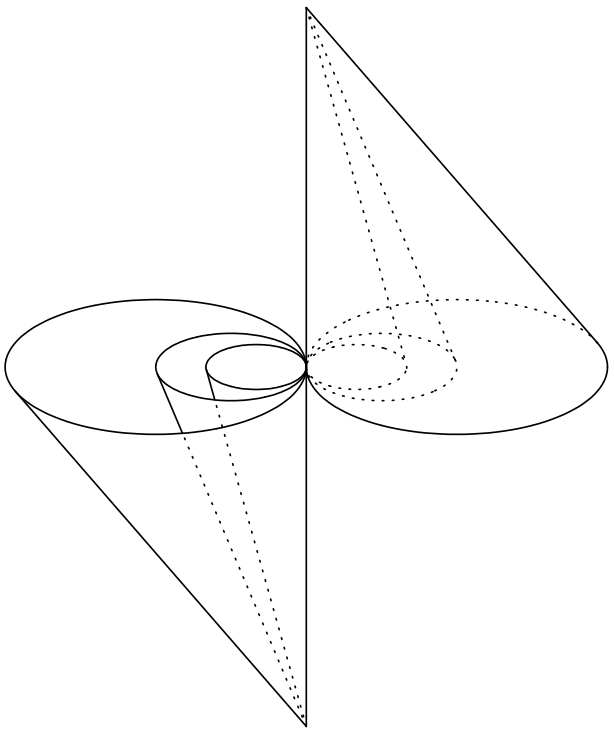

Fig. a (belonging to the Introduction, cf. also 2.13):
This picture shows a one-point union of two coned Hawaiian Earrings and presents them as a three-dimensional example that has phenomena of pathology that in two dimensions cannot occur (cf. 0.7(Fig.c)).

ACKNOWLEDGEMENT: I thank Professor Epstein for suggesting the subject and for making me acquainted with some of the relevant mathematical theories. I also wish to thank Lucien Guillou and Alexis Marin of their interest even in fine points of the proof and asking stimulating questions.

§0. A Quick Trip through Concepts, Terminology and Examples
§1. List of Results
§2. On the proofs of the Theorems as listed in Sect.§1.
§3. Some further aspects of the proofs that have not been mentioned so far

§0. A Quick Trip through Concepts, Terminology and Examples

0.1 The tame Case: The purpose of this paragraph is to remark that a proof of our Main Theorem can easily be obtained for tame subsets of the plane. Here, for example, is a proof when Y is open:

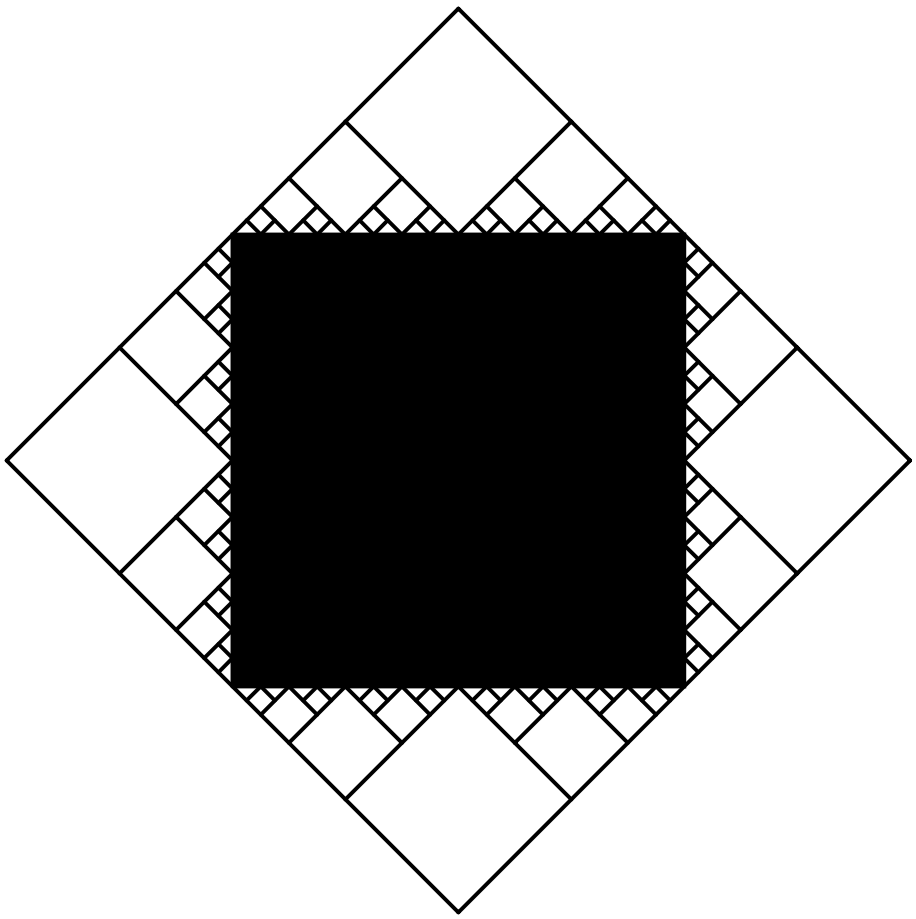

Fig. b (belonging to 0.6):
This picture gives an example for a space which cannot be retracted to a graph-like space that would not have inner points any more.
This is a picture of a compact space of the plane which cannot be retracted to a space with empty interior.

Let $f : S^k \to Y$ be a continuous map for $k \geq 2$. There exists a lift $\widetilde{f} : S^k \to \widetilde{Y}$ to the universal covering of Y. Since Y was assumed to be open, it can be interpreted as a non-compact Riemann surface, and consequently $\widetilde{Y}$ is a Riemann surface with trivial fundamental group. By the Riemann mapping theorem it has to be biholomorphic to $\mathbb{C}$ or to $\mathbb{D}$, and so in any case it is homeomorphic to $\mathbb{R}^2$. Hence there exists a contraction of $\widetilde{f}$ within $\widetilde{Y}$. Composing it with the covering projection $p : \widetilde{Y} \to Y$, we obtain a contraction of f within Y.

0.2 Remark: The above argument cannot be used in general. For example, the Hawaiian Earrings are not semilocally simply connected, and hence they are neither

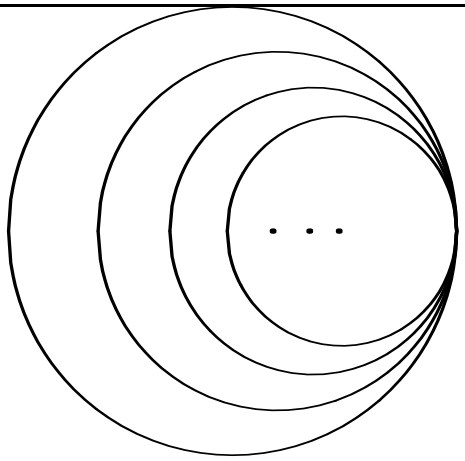

Fig. c (belonging to 0.7(3.)):
This picture shows a topological space named the "Hawaiian Earrings".

homotopy equivalent to an open subset of the plane, nor does the classical theory permit the construction of a universal covering space. Thus we need to construct tools which will replace the use of covering theory and of the Riemann Mapping Theorem in non-tame cases. For that purposes we will make use of the following term which is more common in combinatorial group theory than in topology (cf. [ECHLPT].(Def.3.6.1)):

0.3 Definition:

(i) Let X be a pathwise connected space with a fixed base point x_0. Further fix for every $x \in X$ one path $c_x : [0,1] \to X$ satisfying $c_x(0) = x_0$ and $c_x(1) = x$. Such a system of paths, bijectively associated in this way with the points of X, is called a *"combing"* of X. However the concept of a combing needs some modification to be used in our paper:

(ii) A *"homotopy combing"* of a path-connected space X with a base point x_0 associates to each point $x \in X$ a system of paths containing one path in each relative homotopy class of paths between x and x_0.

Remark: The existence of a homotopy combing is trivial for each pathwise connected space, obtaining additional properties which make the concept useful might be very non-trivial.

0.4 On how the concept of a homotopy combing is used to prove Theorem A:

A homotopy combing gives us a way of attempting to construct an arbitrary continuous contraction map for a sphere: If we assume that the preimage $S^k \quad (k \geq 2)$ of our sphere also has a fixed base point z_0 which is mapped by our continuous mapping f to the base point y_0 of Y, then each image point $y = f(z)$ of our sphere defines a relative homotopy class $\alpha(z)$, namely the image of any path joining z_0 to z in S^k. Hence the task of contracting $f(S^k)$ could be done by sliding each point $f(z)$ down the corresponding path of the homotopy combing. In formulae: If $H : S^k \times I \longrightarrow Y$ denotes the contracting

Area belonging to some hole

Area belonging to a horizontal corridor

Area belonging to a vertical corridor

Crossing region of corridors, i.e. area belonging to both types of corridors

Boundary line of a hole of the i^{th} corridor lattice

Boundary line of a corridor of the i^{th} lattice

Boundary line of a hole of the $(i+1)^{\text{st}}$ lattice

Boundary line of a corridor of the $(i+1)^{\text{st}}$ lattice

Boundary line of a hole of the $(i+2)^{\text{nd}}$ lattice

Boundary line of a corridor of the $(i+2)^{\text{nd}}$ lattice

Fig. $\mathbf{d_0}$ (belonging to 0.7(4.)):
The four main pictures belonging to the series of Fig.d-drawings (Fig.d_1–Fig.d_4) are devoted to showing according to the same scale adjacent corridor lattices of some part of our space X. This figure provides a key to Fig.d_1–Fig.d_6.

homotopy then

$$H(z,t) := c_{f(z),\alpha(z)}(t), \tag{1}$$

where $(c_{y,\alpha})_{y,\alpha}$ denotes our homotopy combing. This sliding process will only be continuous, when our homotopy combing satisfies according properties. To state them precisely, we introduce the following terminology:

(i) Let α_i be a relative homotopy class of paths that connects the base point y_0 to the point P_i. Then we call

> $\inf\{\text{diam}(U) \mid U \subset Y$, U contains a path $u : [0,1] \to Y$ with $u(0) = P_1$ and $u(1) = P_2$ such that $u_1 \cdot u \cdot u_2 \simeq 0$ in Y, where where u_1 represents α_1 and u_2 with reversed orientation represents $\alpha_2\}$.

the *"homotopic distance"* between the homotopy classes α_1 and α_2 and denote it by $d((P_1,\alpha_1),(P_2,\alpha_2))$ (cf. 2.13(ii)).

(ii) $\widehat{Y_{y_0}} := \{(P,\alpha) \mid P \in Y$, α is a relative homotopy classes of paths that connects y_0 to $P\}$. With the homotopic distance as defined in (i), $\widehat{Y_{y_0}}$ becomes a metric space (cf. 2.13(ii)).

(iii) $\widetilde{\mathcal{L}_{y_0}}(Y)$ is the space of all parametrized paths in Y that start at the base point y_0.

(iv) $\mathcal{L}_{y_0}(Y)$ is the analogous set of all abstract paths ("abstract" = only defined up to monotonic reparametrization).

Observe that a homotopy combing can be understood as a mapping $\widehat{Y_{y_0}} \to \mathcal{L}_{y_0}(Y)$ or $\widehat{Y_{y_0}} \to \widetilde{\mathcal{L}_{y_0}}(Y)$, depending on whether the combing paths are regarded as given with a concrete parametrization or as abstract paths, only. Essentially our Theorem A will follow by constructing $(c_{y,\alpha})_{y,\alpha}$ as such a parametrized homotopy combing that gives a continuous mapping $\widehat{Y_{y_0}} \to \widetilde{\mathcal{L}_{y_0}}(Y)$, where $\widetilde{\mathcal{L}_{y_0}}(Y)$ is metricized by the supremum metric. Since by the arguments as given in the first paragraph of 0.4 any basepoint-compatible mapping $f : S^k \to Y$ can continuously be lifted to a map $\widetilde{f} : S^k \to \widehat{Y_{y_0}}$, by constructing the composition $(c_{y,\alpha})_{y,\alpha} \circ \widetilde{f}$ we can associate a continuous map $S^k \to \widetilde{\mathcal{L}_{y_0}}(Y)$ with each given map $f : S^k \to Y$. By elementary properties of a path space like $\widetilde{\mathcal{L}_{y_0}}(Y)$, the composition $(c_{y,\alpha})_{y,\alpha} \circ \widetilde{f}$ has a natural contraction. Our homotopy H results from naturally projecting this contraction back into the space Y. Apart from the definition of the continuous parametrized homotopy combing $(c_{y,\alpha})_{y,\alpha}$ all these constructions are straightforward and not to be described in detail in this report. Our homotopy combing now will be constructed in two steps, namely

(1.) An abstract homotopy combing $\widehat{Y_{y_0}} \to \mathcal{L}_{y_0}(Y)$ will be constructed first (cf. 2.18).

(2.) This abstract homotopy combing will then be lifted to a parametrized homotopy combing $\widehat{Y_{y_0}} \to \widetilde{\mathcal{L}_{y_0}}(Y)$ (cf. 2.20).

Since there does not seem to exist a suitable metric on $\mathcal{L}_{y_0}(Y)$ whose consideration could ease the discussion whether or not our (1.)-homotopy combing can be lifted in the sense of (2.), we will not introduce any metric on $\mathcal{L}_{y_0}(Y)$. Consequently we will not be able to discuss continuity of our (1.)-homotopy combing. Although the existence of a (1.)-homotopy combing is trivial by Zorn's Lemma, it will take more sophisticated arguments to find such a (1.)-homotopy combing that actually permits the lifting in the sense of (2.) to a continuous map, see below.

0.5 Remark: In order to construct a homotopy combing in the sense of 0.3(ii) for possibly very pathological spaces, we plan to tackle pathology in two ways:

(i) First, by showing that we can replace the subspace Y of the plane by a nicer space X without loss of generality (cf. 0.7(1.)–(2.), Thm.1.2)— and

(ii) second, by describing the topology and the variety of paths of X combinatorially (cf. Theorems 1.7 and 1.9).

The combinatorial objects that will be constructed for the purposes of (ii) are called *"word sequences"* (cf. 0.7(3.)–6)). They will allow to select combing paths out of the variety of all paths, and they will provide such an amount of information that permits to parametrize the selected paths and hence to lift the combing in the sense of 0.4(2.). That way we will also be able to prove continuity of that lift (0.4(iii)–(iv), Theorem 1.10).

0.6 Remark: In order to define the parametrization that provides the lifting of our homotopy combing (cf. 0.4(2.)), the most natural idea, a constant speed parametrization, cannot be used since for wild spaces there exist homotopy classes where all representatives are not rectifiable. Also note, that wild subsets of the plane cannot, as tame ones, always be retracted to spaces with empty interior, an example is given below:

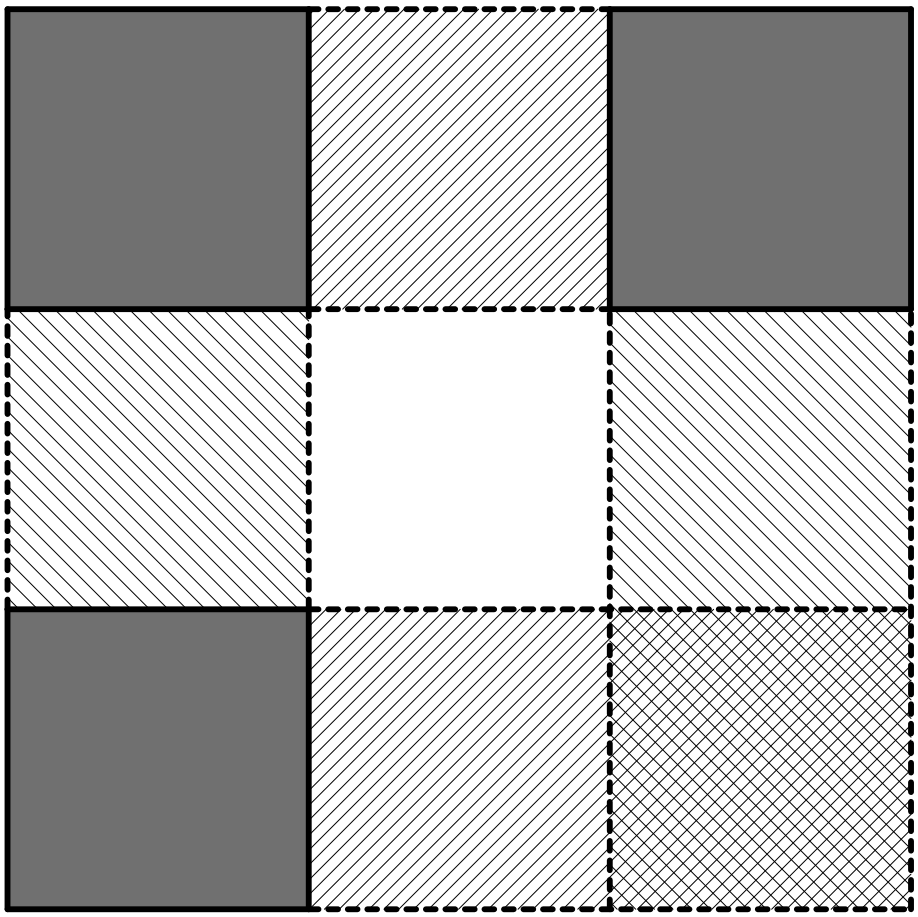

Fig. $\mathbf{d_1}$ (belonging to 0.7(4.)):
This figure pictures the corridors of a fixed (say i^{th}) lattice within this part and the holes as far as they have been integrated into that lattice.

Consider the space Z which is essentially given as a subset of the plane by the points that are painted black in Fig.b. More precisely, Z consists of a solidly filled square in the middle, and of four isosceles right-angled triangles that are only filled by a graph-like pattern of lines. The pattern of lines is constructed by an infinite iterative process, and, of course, only the first finitely many iteration steps are painted in Fig.b.
Now assume that we select two points at opposite boundaries of the inner square and choose inside the neighbourhood of each of these points Hawaiian Earrings, i.e. a subspace of the outer triangle which is homotopy equivalent to the space as pictured in 0.7(Fig.c). If we try to get rid of the inner points of Z by collapsing the inner square to a single point, it would be possible to define a continuous path which runs over all of these loops of the Hawaiian Earrings alternately. Such a path could not have been con-

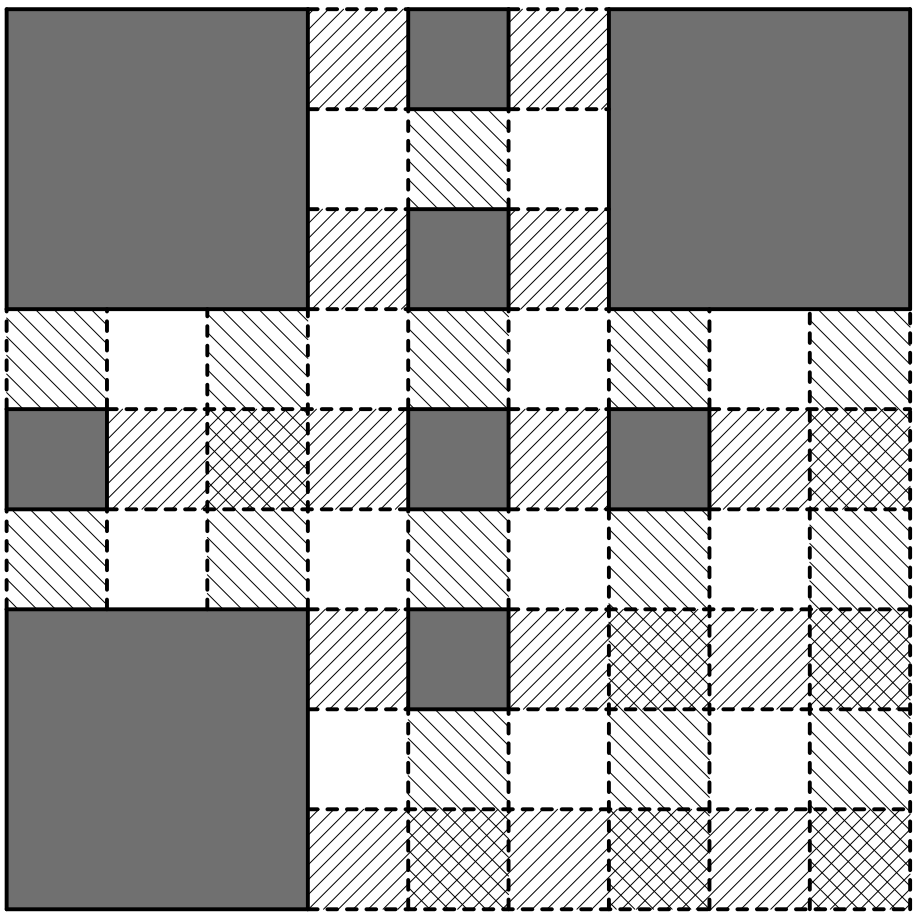

Fig. d_2 (belonging to 0.7(4.)):
This drawing covers the same part of X as Fig.d_1. It shows the corridors of the $(i+1)^{st}$ lattice and the corresponding holes.

structed before the collaps of that inner square, and hence this collaps is not a homotopy equivalence.

0.7 The basic ideas and concepts for the proof of Theorem A:

(1.) The first step is to note that we can assume that our space Y is the continuous image of a sphere. Such an image is compact, connected, and locally pathwise connected, see [Will].(Thm.31.1&31.4–5). If we manage to contract our sphere within its image domain, we are done. In is some case we do not, we can certainly not do it for all spaces.

(2.) Let X be a compact, connected and locally path-connected subset of S^2 and let U be a component of the complement. It is not that hard to see that topologically U must be a disk (cf. [Zas0].A.2.5). Then there is a holomorphic isomorphism from the

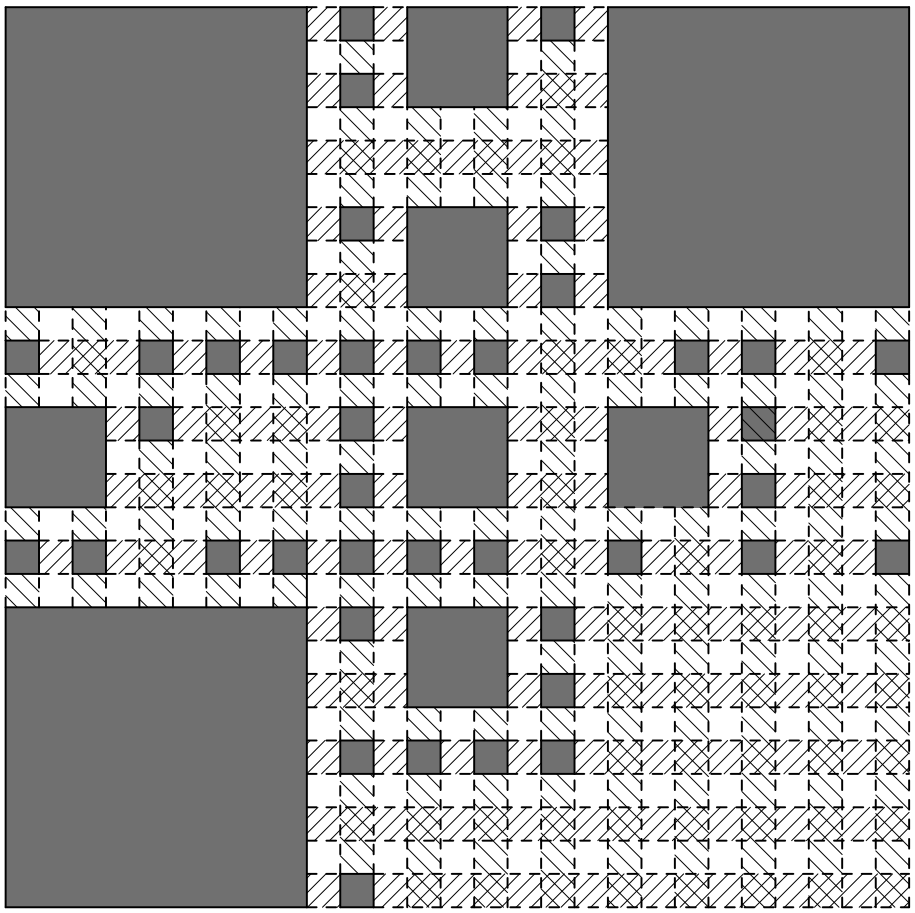

Fig. d3 (belonging to 0.7(4.)):
This drawing shows how the $(i+2)^{\text{nd}}$ corridor lattice might look like in the same region that has been pictured in Fig.d_1 and Fig.d_2 already.

open unit disk to U, that by a Theorem of POMMERENKE (cf. [Pomk].(Thm.2.1)) extends to a continuous map $\mathbb{D} \to \overline{U}$. Thus we are able to retract $\overline{U} - \{P\} \to \overline{U} - U$. By using such retractions we can replace any odd-shaped hole by a nice-shaped one. I.e. instead of working with the space originally given by the continuous image of our sphere, we will define a nice-shaped disk inside each complementary complement, and enlarge X to be the union of those disks. This larger space deformation retracts to the original version of X (cf. Theorem 1.2).

(3.) Another ingredient for our paper, apart from results on the boundary behaviour of conformal mappings as described above, is an adaptation of the theory of word sequences as developed for the fundamental group of the Hawaiian Earrings (cf. [Zas2].§2, [Zas]). The Hawaiian Earrings (cf. Fig.c) can in this context be un-

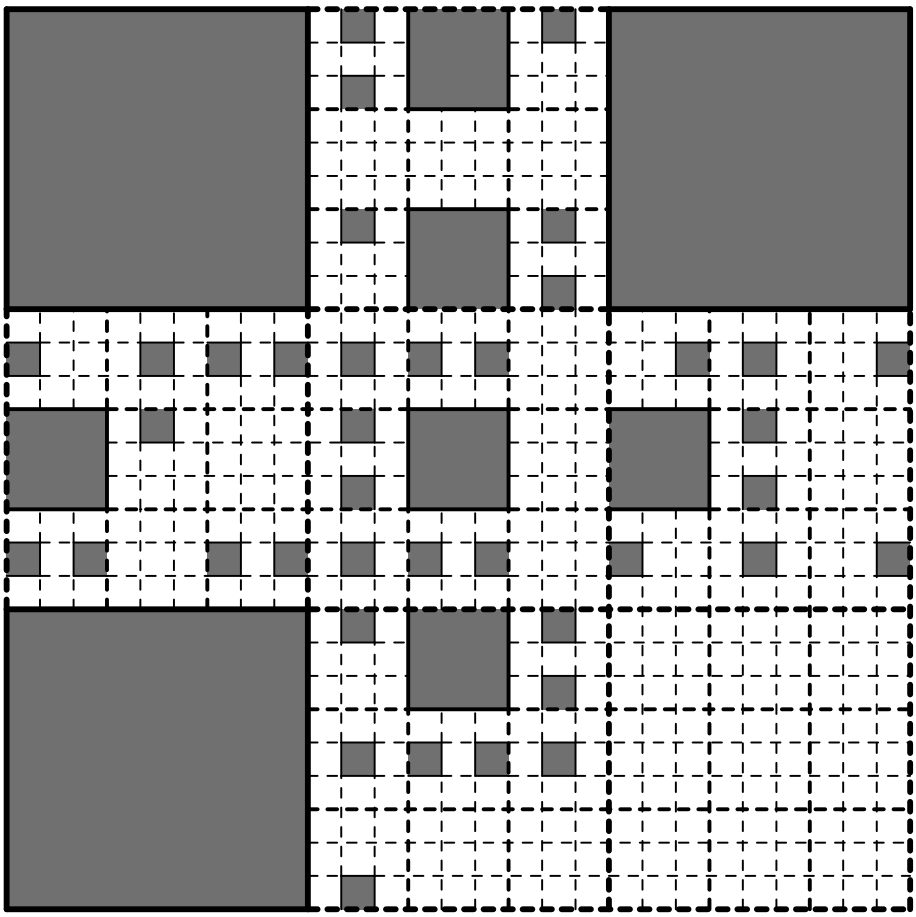

Fig. d_4 (belonging to 0.7(4.)):
This drawing tries to give some impression of the the entire corridor network. For that purpose the three corridor lattices as they have been pictured in Fig.d_1–Fig.d_3 have been drawn into this figure simultaneously with the hatching of the corridors being removed.

derstood as one of the simplest subspaces of the Euclidean plane that already has some accumulation phenomena. Their fundamental group has been under investigation since the fifties (e.g. [Gr2].(Thm.6.3), [Higm].(Proof of Thm.6), [MM].(Thm.4.1), [dSm].§3, [Eda1]) and has recently been rediscovered and inspired two working groups (SIERADSKI & BOGLEY and CANNON & CONNER, cf. [B–S], [C–C]) to generalizing to notion of (freeness of) groups. Observe that Eda is working with yet another concept independently developed a couple of years before ([Eda2], [E–K]). In [Zas2].§2 the classical concept of describing the fundamental group of the Hawaiian Earrings as a subgroup of $\varprojlim(F_m)$ where $F_m = \langle \alpha_1, \ldots, \alpha_m \mid \ - \ \rangle$, has been adapted to some concept of word sequences, and in the following this concept of word

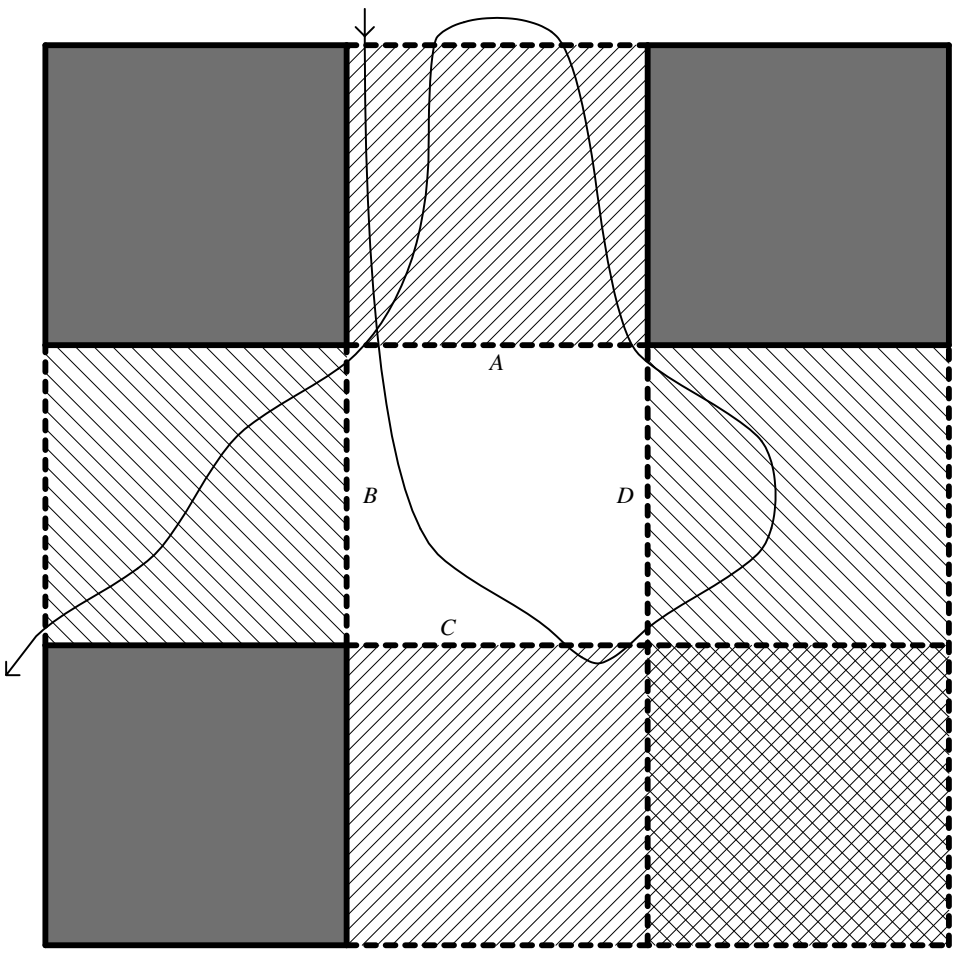

Fig. d_5 (belonging to (4.)):
This diagram shows again the same lattices as Fig.d_1 and, in addition, demonstrates how a word (which appears in the upper margin) must be read from a path that is running through the i^{th} corridor lattice. For this purpose names have been attached to our corridors.

sequences will be replaced by a more sophisticated concept that can cope with more than one accumulation point of holes.

(4.) Roughly speaking, here in this paper our concept of word sequences will be based on regarding the area that is placed between two neighbouring holes as a *"corridor"* (cf. Fig.d_1–Fig.d_3) and on defining a word sequence to be a record of which corridors a path is crossing in which order. An example for a part of a word sequence is given in Fig.d_5–Fig.d_6: Here the essential drawings from Fig.d_1–Fig.d_3 have been copied, but now names have been attached to each of these corridors and a path u has been drawn into the corridor lattices. Let ω be the word sequence that is associated with u. In the margin of Figures d_5 and d_6 we find the record of acts of crossing corridors

as it performed by the path in these figure. These words must be understood as those parts of the words ω_i and ω_{i+1} of ω, respectively, that are associated with the visible segment of u. Observe that when in ω_i some letter, say A^{-1}, gives the information that the associated path crosses some corridor, then "A^{-1}" cannot be found in ω_{i+1} any more. To wit, on this level "A^{-1}" is "substituted" by at least two finer letters (here "$a_1^{-1}a_3^{-1}$") that report acts of crossing finer corridors that on this level are drawn into the area of the corridor A. Hence the essential difference between the [Zas2]-word sequences and our word sequences can be briefly summed up as follows: While the [Zas2]-word sequences have an insertion structure, our word sequences now have a *"substitution structure"*. In that context observe that, if on the i^{th} lattice a path goes through a crossing region of corridors and hence is crossing two corridors simultaneously, then, while on the i^{th} lattice the corresponding letters must come one after the other, their substitutes on the $(i+1)^{\text{st}}$ lattice may show up in a mixed order!

(5.) *The outline of our proof is as follows:* By using the methods as described in (2.) we will show that we can reduce the situation to a space where all the hole are arranged according to a certain pattern (cf. Fig.d/Theorem 1.2). For such spaces a corridor network can be defined and hence a concept of word sequences can be set up. It can be understood on a mere combinatorial basis what combination of symbols (each symbol will be associated to a corridor and to the direction it is crossed by a path) gives valid word sequences (i.e. such word sequences that actually correspond to paths, cf. Thm.1.4), when two of those sequences describe the same path (cf. Theorem 1.7(i)), and when two of those sequences describe relatively homotopic paths (cf. Theorem 1.7(ii)). Having access to such a mere combinatorial description will allow us to select such word sequences that are reduced in a certain sense (cf. Theorem 1.9), and the reduced word sequences will give combings paths of an (abstract, but liftable, cf. 0.4(1.)–(2.)) homotopy combing in the sense of 0.4(iii)–(iv). Thus using such a combinatorial method may be understood as choosing "algebraically shortest paths" where due to the phenomena of non-rectifiable paths "geometrically shortest paths" cannot be defined in a natural way.

(6.) The "pattern" as firstly mentioned at the beginning of (5.) will require that all holes come in a discontinuous set of sizes, and that the widths of corridors are restricted to the same set of sizes. When considering the holes and the corridors of a certain width only, we will talk of a *"corridor lattice"*, and we will reserve the word *"network"* for considering all holes and corridors of all sizes. If $\omega = (\omega_i)_{i\in\mathbb{N}}$ is a word sequence in the sense of (4.), then ω_i will be a record of acts of crossing of all those corridors of a certain width only that is related to the index i ("of the i^{th} corridor lattice"). Consequently the index of a lattice or of some word taken from a word sequences may be understood as the *"level* of approximation". In that sense from the combinatorial information given by the word ω_i, a certain approximation to our given path may be constructed, namely a path that runs through the same regions bounded by adjacent corridors of the i^{th} lattice and performs the same acts of crossing those corridors as the path to be approximated, and hence is never further apart than the mesh-width of the i^{th} lattice as the path to be approximated. A better approximation of our path may be reconstructed from words

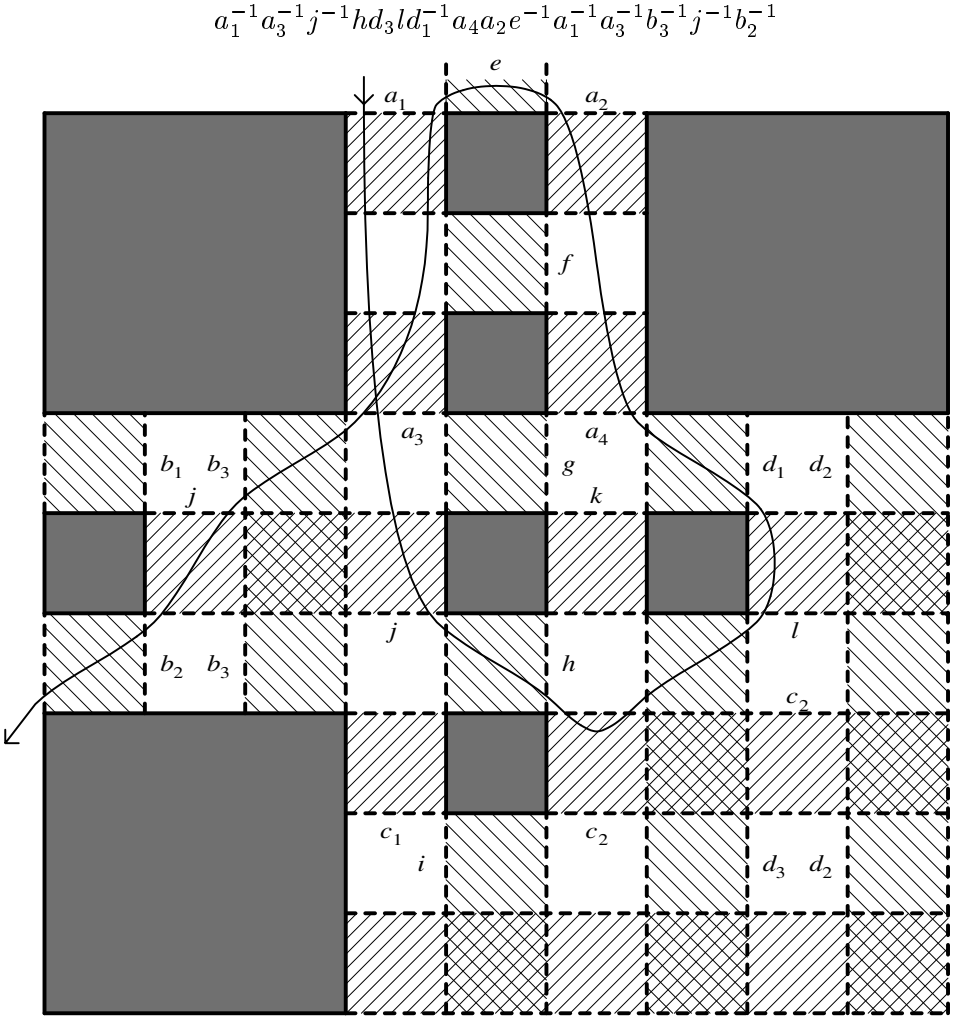

Fig. d6: (belonging to (4.)):
This diagram shows how the same path that was visible in Fig.d5 appears in the $(i+1)^{st}$ lattice. Now in this lattice a word is read from our path. The words that have been derived in Fig.d5 and in this figure together show how two adjacent entries of a word sequence describing our path may look like.

ω_j with $j > i$, and it will turn out (cf. 1.6) that up to monotonic reparametrization a given path may be reconstructed from the overall information that is contained in the entire word sequence ω.

The preceding subparagraph explained the geometric idea behind the *"realization mapping* Φ" that finds a path for a given word sequence. (cf. 1.6/2.2). Observe that such a construction can only work and end with a well-defined mapping if the following two conditions are satisfied:

(i) We need a full network with horizontal and with vertical corridors so that we can make sure that the x- and the y-coordinates of the approximating paths are close to the path to be approximated. Both coordinates need to be controlled if a

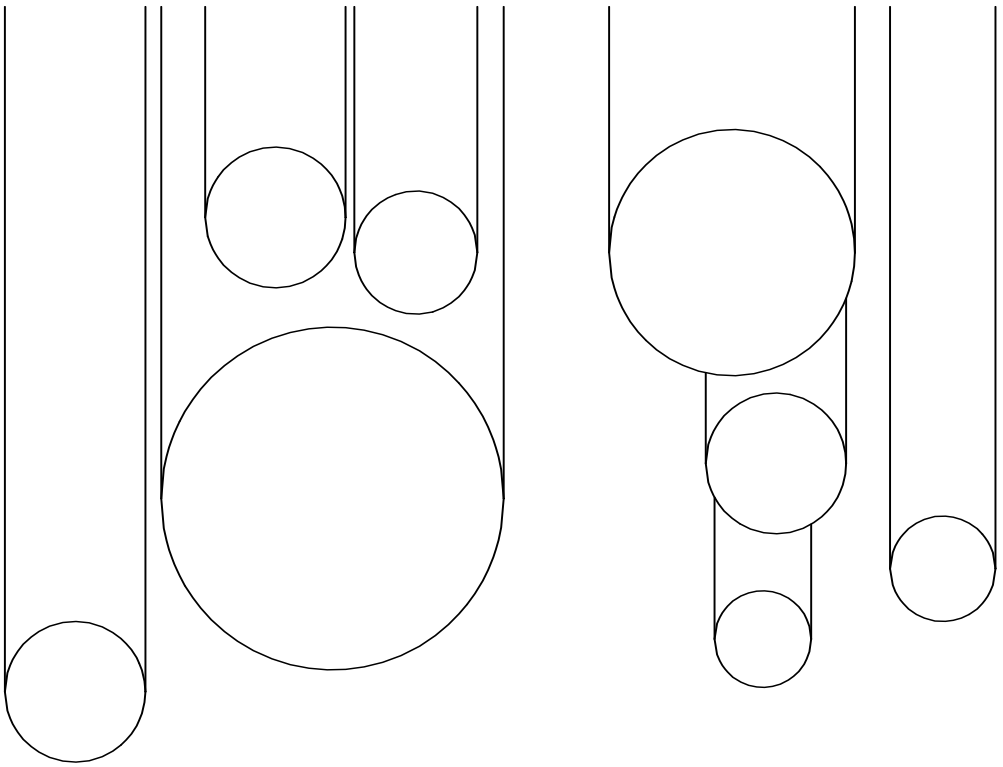

Fig.e (belonging to (6.)):
This figure pictures the probably easiest concept of holes brought into a nice shape and corridors attached to it: Each hole has been given a round shape and has been connected to infinity by precisely one corridor. We call such a system of corridors a "slice system". Despite we are unable to read word sequences from such slice systems, for some conclusions such slice systems need to be considered besides the full networks as pictured in Fig.d.

construction as described above shall in any case give a uniform Cauchy-sequence of approximating paths, and we need this uniformity to conclude that there exists a limit-object, i.e. that Φ can actually be defined to associate this limit-path to any valid word sequence. In a mere slice-system of corridors as it is pictured in Fig.e easily sequences of approximating paths can be constructed that do not converge (cf. 3.1).

(ii) A construction as described above can only work if it is compatible with the convergence on the level of word sequences that is defined by stabilization on each level (otherwise we will not be able to prove that our limit path is actually described by the word sequence that has been used for constructing the approximating paths). This necessity provides an obstruction for defining word sequences uniquely, i.e. we will have to allow the description of paths by more than one word sequence if we are in one of the limit situations as described 2.2(i)–(iii) (cf. 3.2).

(7.) The "pattern" as described in (5.)–(6.) is pictured in Fig.d_0–Fig.d_4. These figures show examples of corridor lattices for three adjacent level indices. Each of the the pictures only shows the holes of our space X that are integrated into the corresponding lattice. If i is the index of the lattice, then X_i denotes the complement of those integrated holes. Each of the X_i-holes is also a hole of X, but X may contain a lot of finer holes, and in figures like Fig.d_1–Fig.d_3 they will only be pictured for higher indices of approximation. Basically we have

$$X_0 \supset X_1 \supset X_2 \supset \ldots\ldots\ , \qquad \bigcap_{i=1}^{\infty} X_i = X. \tag{1}$$

Observe that each of the figures Fig.d_1–Fig.d_3 gives a subdivision of the plane into squares such that by the structure of holes and corridors each of the squares is associated to one of the following attributes *"hole"*, *"vertical corridor"*, *"horizontal corridor"*, *"crossing region of corridors"* and *"pure lattice square"*. The last attribute is defined for those squares to which none of the predecessing attributes can be applied. The letters "H", "v", "h", "c" and "p" are in Formula (2) below used to refer to these attributes, respectively. The class of possible hole- and corridor patterns one example of which is pictured in Fig.d_0–Fig.d_4 can in an inductive way precisely be defined as follows:

Zero-Step X_0 is one square without any subdivision, and this square has the attribute pure lattice square.

Inductive Step Each of the squares X_{i-1} is (3×3)-subdivided for X_i. Depending on the attribute of each of the squares one of the following subdivision schemes is applied:

$$\mathrm{H} \longmapsto \begin{pmatrix} \mathrm{H} & \mathrm{H} & \mathrm{H} \\ \mathrm{H} & \mathrm{H} & \mathrm{H} \\ \mathrm{H} & \mathrm{H} & \mathrm{H} \end{pmatrix} \qquad \mathrm{h} \longmapsto \begin{pmatrix} \mathrm{h} & \vee & \mathrm{h} \\ \mathrm{p} & \mathrm{v} & \mathrm{p} \\ \mathrm{h} & \vee & \mathrm{h} \end{pmatrix}$$

$$\mathrm{p} \longmapsto \begin{pmatrix} \mathrm{p} & \mathrm{v} & \mathrm{p} \\ \mathrm{h} & \vee & \mathrm{h} \\ \mathrm{p} & \mathrm{v} & \mathrm{p} \end{pmatrix} \tag{2}$$

$$\mathrm{v} \longmapsto \begin{pmatrix} \mathrm{v} & \mathrm{p} & \mathrm{v} \\ \vee & \mathrm{h} & \vee \\ \mathrm{v} & \mathrm{p} & \mathrm{v} \end{pmatrix} \qquad \mathrm{c} \longmapsto \begin{pmatrix} \mathrm{c} & \mathrm{h} & \mathrm{c} \\ \mathrm{v} & \mathrm{p} & \mathrm{v} \\ \mathrm{c} & \mathrm{h} & \mathrm{c} \end{pmatrix}$$

The meaning of the letters in this matrices has been explained above, the symbol "$\vee$" is used as an abbreviation for "c or H".

Although it is not visible from (2), it is a demand that holes must never crop up in crossing region of corridors, not even through multiple steps of iteration. On the other hand, the freedom that comes with the $\vee$-entries in the matrices makes sure that spaces with quite a different number of holes and various accumulation phenomena do admit a corridor lattice according to the above definition. This permits to prove an embedding theorem such as Theorem 1.2.

§1. List of Results

In the course of proving the main result of this paper ("Theorem A", cf. the beginning of this paper) some other results had to be proved as well, some of which might have an

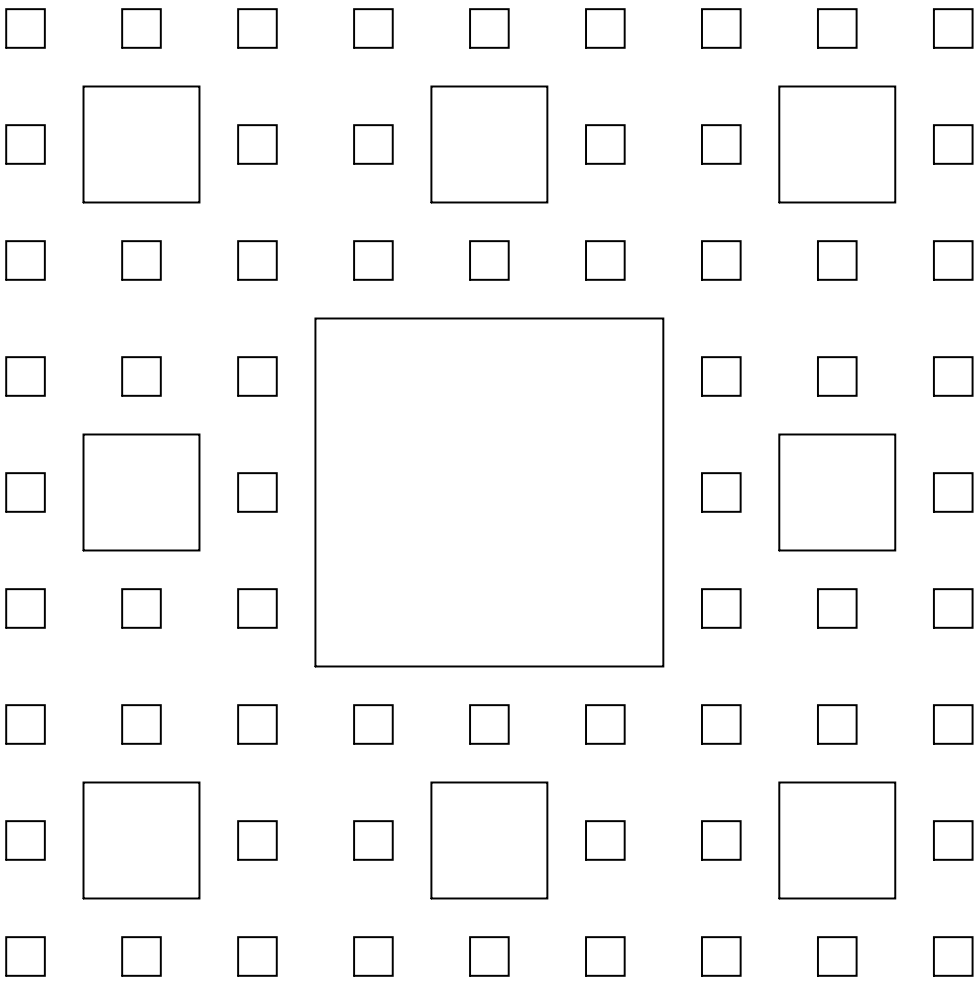

Fig. f (belonging to 1.1):
This picture shows a topological space which is called the "Sierpinski Carpet"
(cf. [Sier], [PJS].§.2.2(Fig.2-20)).

interest of their own. They are listed below, and by doing this we also give an idea how the proof of Theorem A is subdivided into various steps. Section §2 will be devoted to sketching the proofs of the Theorems below and hence to sketching the Proof of Thm.A.

1.1 Definition: The *"Sierpinski Carpet"* is a topological space which can be obtained by taking out holes according to the pattern as pictured in Fig.f. The *"Modified Sierpinski Carpet"* is analogously obtained by taking out open squares according to a different pattern, more precisely: by taking out each of the $\vee$-region according to the inductive scheme as described in 0.7(7.) (cf. Fig.d). Observe that according to the universal property of the Sierpinski Carpet (cf. [Why].(Thm.3&4) and [C–P].(page 36)) the statement of Theorem B remains true if the Sierpinski-Carpet is replaced by the modified one.

1.2 Theorem B: *Any compact and totally and locally pathwise connected subset of* $\mathbb{R}^2$ *can homotopy equivalently be embedded into a space obtained from the Modified Sierpinski carpet by filling an appropriate subset of its holes (cf. 2.4 or [Zas0].A.3.5)).*

1.3 Remark: Although the scheme for constructing a corridor network is fairly rigid, the $\vee$-symbols in Formula 0.7(2) permit for countably many regions either to be a hole of X or to be a region which entirely belongs to X. This gives uncountably many spaces which may have a corridor network as described in 0.7(6.)–(7.). The extremal options for such spaces are the modified Sierpinski Carpet (where each $\vee$-region has become a hole, cf. 1.1), or the entire unit square $[0,1]^2$ where none of the $\vee$-regions has become a hole. From 0.7(Fig.d_1)–(Fig.d_3) it should be clear that with a different X there comes a different system of corridors, and hence, since each corridor is according to 0.7(4.) associated with a symbol, a different *"alphabet"* for defining the set of word sequences for the corridor network of X. Observe that in a rigid definition of the alphabet, the geometrical properties of and relations between corridors (e.g. vertical, horizontal, crossing or not crossing, being contained in each other, being placed in neighbouring regions) are also known on the combinatorial level of the alphabet.

1.4 Theorem: *Let X be a space which is obtained from the modified Sierpinski Carpet by filling a subset of the holes and let $\mathbb{A}$ be the associated alphabet. If we think of the geometrical relations of corridors as encoded into the alphabet-structure as described above, the valid word sequences in the sense of 0.7(5.) can be characterized by a finite number of combinatorial rules.* Henceforth in this paper we will always assume that word sequences are valid in the above sense (cf. 1.5/2.3).

1.5 Remark: In this report we will not in detail present the combinatorial rules for composing word sequences. However, the purpose of this remark is to stress that the origins of these rules can be easily understood by geometry. Essentially we will need two types of rules:

(i) If a path performs a certain act of crossing a corridor, we usually (cf. 3.4) know in which lattice-square the path is in. If the path should not have its endpoint in the lattice square, it has to go out again by crossing one of the four bounding corridors. This geometrical fact gives a restriction on which letters may appear in adjacent positions of a word of a word sequence.

(ii) By looking at the figures 0.7(Fig.d_5) and Fig.d_6 one sees that from the record of the acts of crossing the corridors of the $(i+1)^{\text{st}}$ lattice one can also deduce which corridors of the i^{th} lattice must have been crossed. Up to certain limit cases such as 2.2(ii)–(iii), the i^{th} word of a word sequence may be deduced from the $(i+1)^{\text{st}}$ word. This gives *"compatibility conditions"* that essentially require that the behavior of a path as it follows from the $(i+1)^{\text{st}}$ word of a word sequence must not contradict to the behavior of a path as it follows from the i^{th} word.

1.6 Theorem C: *Let X be a space which is obtained from the Modified Sierpinski Carpet by filling some of its holes, and let $\mathbb{A}$ by the associated alphabet. Then the*

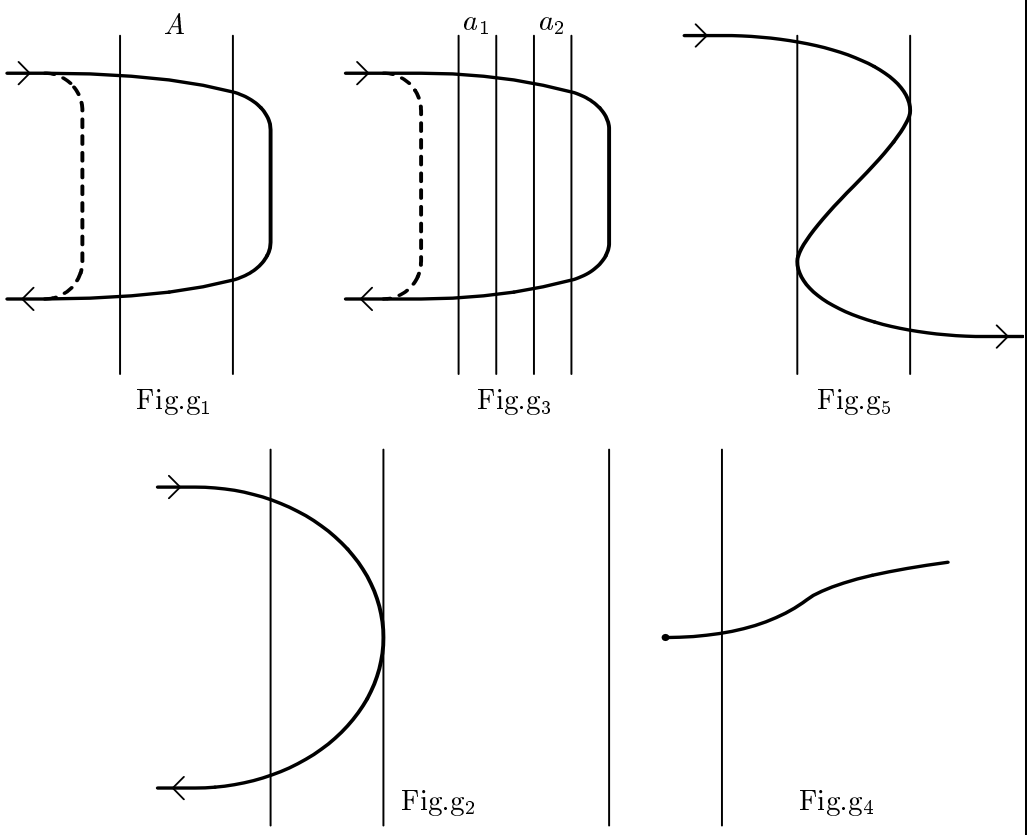

Fig. g (belonging to 1.8 and 2.2):
The purpose of the subfigures of this figure is showing possible behaviors of paths with respect to corridors that give either situations where paths cannot uniquely be described by word sequences, or explain the origins of the transformation processes for word sequences..

natural map as defined in 0.7(6.)

$$\Phi : \{\textit{word sequences modelled on } \mathbb{A}\} \longrightarrow \{\textit{paths in } X\}$$

is onto (cf. 2.5 or [Zas0].C.2.1/[Zas0].C.3.1).

1.7 Theorems D and E:

(i) The fact when two word sequences ω and ω' in the sense of Theorem C describe the same path in X can be combinatorially characterized by finitely many rules. (cf. 2.6 or [Zas0].D.12.3)

(ii) A similar combinatorial description can also be obtained for describing when two word sequences are mapped to relatively homotopic paths (cf. 1.8, 2.17 or [Zas0].D.12.1). The corresponding description differs from the one of (i) just by weakening some of the boundedness conditions (cf. 1.8(3.)).

1.8 Remark: Three different equivalence relations are defined combinatorially in the full version of the paper (cf. [Zas0].B.2.9)), two of which admit the geometrical interpretation as given by the Parts (i) and (ii) of 1.7. More precisely, the *"weak equivalence relation"* and the *"strong equivalence relation"* are to be defined by finitely many combinatorial rules so that Thm.1.7 can be replaced by the statements that the strong/weak equivalence relation correspond to the situation that two word sequences are associated with the same or relatively homotopic paths, respectively. The combinatorial definitions of these equivalence relations will not be given in full detail in this report, but the structure of them and the geometric origin of the corresponding rules will be described below. Each of our three equivalence relations will be described by three types of rules as listed in the following:

Two word sequences ω and ω' are equivalent, if for any two words with the same index

(1.) there exists a finite sequence of elementary processes transforming the i^{th} word of ω into the i^{th} word of ω' *("process demands")*. These elementary processes can either be

(i) *"elementary cancellations"* (or their inverse), i.e. the taking out (or inserting) of two adjacent letters which are inverse to each other. The origin is given by relative homotopies as drawn in Fig.g_1. The second type of processes are

(ii) *"elementary transpositions"*, which means interchanging two adjacent letters if they belong to commuting variables. In this context observe that *"commuting"* is an attribute which is associated precisely to those pairs of variables of letters where one belongs to a horizontal corridor and the other to a vertical corridor and where the two corridors cross each other in a crossing region. Relative homotopies that correspond to elementary transpositions are pictured in 2.7(Fig.j). — The third type of elementary processes are

(iii) *"elementary truncations"* or *"appendances"*. The geometric origin for this process comes from the necessity that in order to establish a concept of word sequences where the concatenation of paths is compatible with the concept of concatenating word sequences, it is necessary to treat situations as they are pictured in Fig.g_4 as a limit case. In particular the situation where a path goes into a corridor and ends in the interior must be treated as an incompleted act of crossing, which can either be a limit case of a completed act of crossing or of not having gone into the corridor at all. Consequently we are as well allowed to put or not to put the corresponding letter into our word sequences, and we need elementary truncations and appendances for being able to switch between these two possibilities. In principle these processes allow to delete the first or the last letter of a word or to append one letter at either of the ends of one word, but confer the remark after (3.).

(2.) In addition to (1.), this sequence of finite sequences that changes the words of ω into those of ω' as described above has to be *"compatible"*, i.e. the processes that are applied to the i^{th} words of our word sequence must induce the processes that have to be applied to the $(i-1)^{\text{th}}$ words. The geometrical origin of these compatibility demands is the same as in 1.5(ii): Whenever a valid sequence of transformation processes describes some moving around of a path with respect to the i^{th} corridor lattice, then

the moves that are described for the other lattices must not contradict to those that given on the i^{th} level.

(3.) "*Boundedness Demands*". Consider Fig.g_3 which shows the same process as Fig.g_1 for the next finer level. On that level we have transform $a_1 a_2 a_2^{-1} a_1^{-1}$ into the empty word by firstly cancelling a_2 against a_2^{-1} and then cancelling a_1 against a_1^{-1}. We call this a "nested pair of cancellations". When considering more finer levels, we see that the same geometric process needs to be described by nested cancellations with a larger nesting depth and that with respect to the level index the nesting depth is at least as fast growing as the powers of two. Similar nesting depths can also be defined for transpositions and truncations. Boundedness conditions now require that the nesting depth n_i that for each level index i separately is computed results in a bounded sequence $(n_i)_{i\in\mathbb{N}}$. Weak and strong equivalence will differ by the types of transformations that have to be taken into account when computing the n_i-values. It is not so hard to conclude that imposing a certain boundedness condition implies that a path cannot be moved around by the corresponding processes at all, since the distance for moving it with respect to the i^{th} level is given by the product of the bound and the mesh-width of the i^{th} corridor lattice. And since the mesh-width is a null-sequence, by the demand of compatibility of the processes for all lattices we get that the path cannot be moved around at all.

In our paper bounded processes are all used to switch between alternative descriptions of word sequences, as they occur in certain limit cases (cf. Fig.g_2). According to these principles it should not be surprising that for strong equivalence (cf. Thm.D) one needs to impose boundedness conditions to all types of processes. For the weak homotopy equivalence that corresponds to Thm.E we impose a boundedness condition only for the truncation processes. This makes sure that Process (iii) cannot be used to cut off an initial or a terminal segment of a path and hence ensures that only relative homotopies are described by the weak equivalence relation.

Finally, we also mention that the third equivalence relation as needed in this paper is the so-called "firm" one. *Firm equivalence* is defined by letting transpositions be the only valid elementary processes and applying boundedness conditions to them.

1.9 Theorem F: *By subdividing the equivalence transformations as defined in 1.8(1.) into shortening and enlarging ones, the natural concept of reducible and irreducible is established for word sequences. According to the weak equivalence relation, each word sequence is equivalent to an irreducible one, and, in addition, the irreducible ones of any weak equivalence class are firmly equivalent* (cf. 2.15 or [Zas0].D.7.10).

Theorem F can be understood as having characterized "algebraically shortest paths" within each relative homotopy class of paths in a space obtained from the Modified Sierpinski carpet by filling in an arbitrary subpattern of its holes (cf. 0.7(5.))

1.10 Theorem G: *Let X be a space obtained from the modified Sierpinski Carpet by filling an arbitrary subpattern of holes, and for $x_0 = (0,0) \in X$ let $\widetilde{X_{x_0}}$ and $\widetilde{\mathcal{L}_{x_0}}(X)$ be*

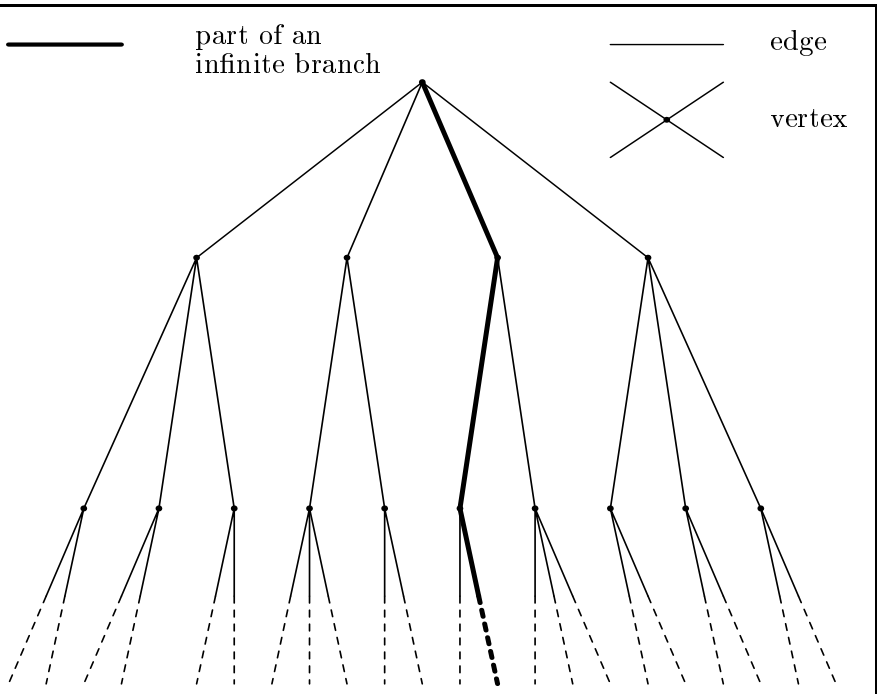

Fig. $\mathbf{h_1}$ (belonging to the 2.1):
The picture shows the standard type of tree to which the Königslemma-type of conclusions can be applied.

as defined in 0.4(iii)/(iv). Then there exists a continuous section $\widehat{X_{x_0}} \to \widetilde{\mathcal{L}_{x_0}X}$ *(cf. 2.18 or [Zas0].B.3.2).*

§2. On the proofs of the Theorems as listed in Sect.§1.

2.1 On a graph-theoretical conclusion following König:

A simple graph-theoretical observation which is due to Dénes KÖNIG (cf. [Kö1].§3, [Kö2].VI.§2(Satz 6) is at various places useful for our proofs. König observed, that (in modern language) a finitely branching infinite tree has to have an infinite branch (cf. Fig.h). There are only finitely many edges emanating from the root vertex, and so at least one of them has to lead to infinitely many vertices in our tree. Such an edge may be used as the first one in the infinite branch to be constructed, and by iterating this construction principle all edges of our infinite branch can be found. In several contexts of the proofs of the Theorems of Sect.§1 it is often vital to know that the compatibility demands (1.5(ii) and 1.8(2.)) can be in a graph-theoretical way expressed by structures of finitely branching infinite trees or by similar graphs. Since infinite branches correspond to compatible patterns, in [Zas0] at various places where by the original construction compatibility has not been ensured, an application of the Königslemma

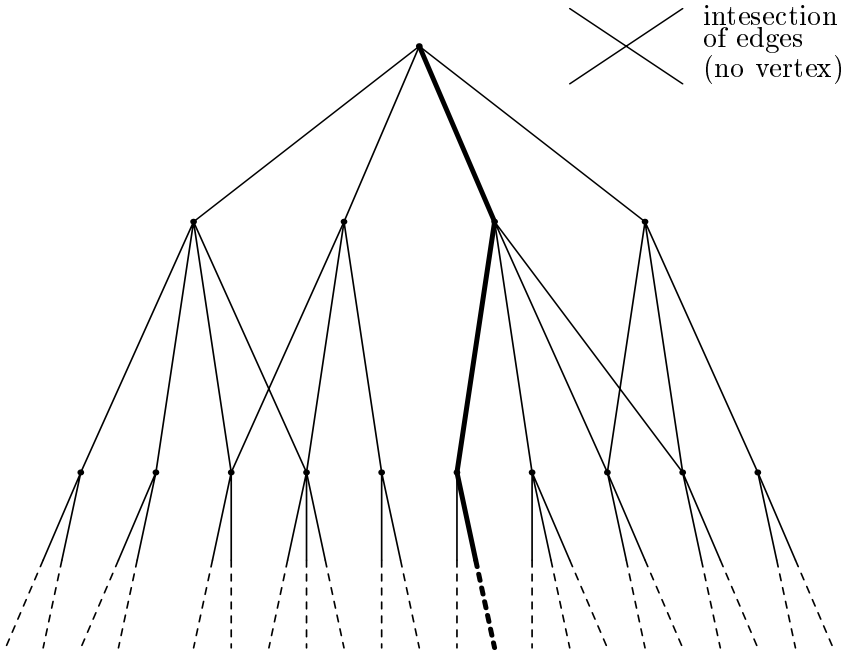

Fig. h_2 (belonging to the 2.1):
This picture shows a more general type of graph than that as seen in Fig.h_1 to which also the Königslemmatype of conclusions can be applied. The picture shows the standard type of tree to which the Königslemma-type of conclusions can be applied.

proves the existence of a compatible pattern.

2.2 On the reading process:

The purpose of this remark is to note that the map Φ as described in 0.7(6.)/1.6 has an inverse Ψ which is also combinatorially discussed in the fully expanded version [Zas0]. Since Φ is onto, but not one-to-one, Ψ can only be established as a multivalued map. It is called *"reading* (a word sequence from a given path)", with respect to *"realizing* (a word sequence by a path)", as we will call Φ. The reading process has essentially been motivated by 0.7(Fig.d_5)–0.7(Fig.d_6), in order to find the i^{th} word of a word sequence one has to make a record which corridors of the i^{th} lattice are crossed in which order by our path. In that context a path $u : [0,1] \to X$ is said *"to cross a corridor* in an interval $[a,b]$", if $u([a,b))$ takes on only values inside or on the boundary of a corridor, and if $u(a)$ and $u(b)$ are values on opposite boundary lines. The reading process is multivalued, since in certain limit cases we are given the freedom of adjusting word sequences in different ways (cf. 0.7(ii)). The geometric origins of these limit cases are as follows :

(i) If a path is about to cross a corridor, touches the opposite boundary, but then turns around without having intersected that boundary, then we can or we cannot regard

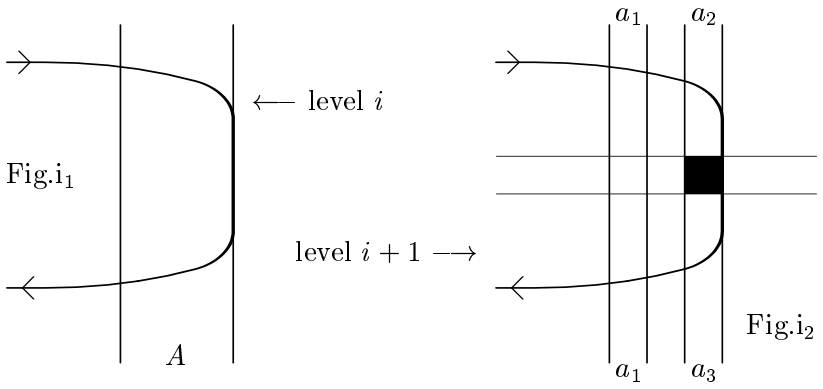

Fig. i: (beloging to 2.2)
Observe that according to (i) on the i^{th} level it appears that AA^{-1} and the empty string are equally valid readings for the pictured segment of our path. However, on the $(i+1)^{\text{st}}$ level $a_1a_2a_3^{-1}a_1^{-1}$ is the only valid reading, and since "a_1a_2" and "$a_3^{-1}a_1^{-1}$" seem to indicate complete acts of crossing on the i^{th} level, we note that for the i^{th} level only the first reading permits to be compatibly extended throughout the word sequence.

the corridor as being crossed (cf. 1.8(Fig.g$_2$)).

(ii) If a path goes all the way diagonal through a crossing region of corridors (which implies that it crosses two corridors simultaneously), we will have to make an arbitrary decision which of the corresponding variables is to be put at first place. However note, that this is not the only boundary case in this context. Because, whenever a path only touches a single point of such a crossing region and performs a crossing of the two corridors, it could be approximated from both sides, and hence all these cases are to be treated as cases where we have non-unique word sequences.

(iii) The third limit case is the situation where a path ends in the interior of a corridor. As already explained in 1.8(iii), we are in that case the same way allowed to treat or not to treat a corridor as being crossed.

A more careful description of the reading processes would it also make necessary to set up rules how the freedom that is described in (i)–(ii) must be used so as not to destroy compatibility. Also, in a fully expanded version one has to cope with the phenomena of non-extendable readings. These are situations where the behavior of our path with respect to the i^{th} corridor lattice only permits such readings which are not compatible with the reading as chosen for the predecessing levels. Fortunately, a careful discussion gives that this case is essentially reduced to the situation as described in Fig.i.

2.3 On the combinatorial characterization of word sequences (cf. 1.4):
The geometric pictures that need to be analyzed to prove Theorem 1.4 have already been described in 1.5. Recall that it was by 0.7(4.) the basic idea of a word sequence to acknowledge each act of crossing a corridor by a letter. I.e. if a path only goes a bit into a corridor from one side, but then turns around and leaves it on the same side, this

is not to be acknowledged by a letter. Since for corridors of a fixed width a path can for continuity reasons only perform a finite number of acts of crossing, this makes sure that each word sequence is finite on each of its levels. In addition, once having reduced our situation on the combinatorial level to the question, which corridors have been crossed and hence in which lattice square our path is in, there are only finitely many possibilities which act of crossing, if seen on the combinatorial level, our path could do next. This is the essential observation for the finiteness as claimed by Theorem 1.4.
This finiteness is not overturned if one takes into account that by 2.2(i)–(iii) it is often necessary to treat limit-cases separately. Phenomena such as constant paths (cf. 3.3), paths that perform a simultaneous tangency to a horizontal and a vertical line or that are coinciding for a segment with a boundary line, or paths having its endpoint on a boundary line or even on a corner point give at the end much more cases that need to be discussed as it might have been the first guess.
Finiteness does also hold if one simultaneously considers a picture with the corridor lines of the i^{th} and of the $(i+1)^{\text{st}}$ lattice and discusses the crossing behavior of a path with respect to these two pictures. From the discussion of the underlying combinatorial possible behavior of a path with respect to such a picture, it follows that also 1.5(ii) can be expressed in a finite number of rules.

2.4 On the Proof of Theorem B (cf. 1.2):
The basic tool for proving this theorem is the freedom to retract any hole, i.e. any complementary region of a compact, path-connected and locally path-connected subset of the plane in a homotopy equivalent way to a square-shaped hole (as already described in 0.7(2.)). Theorem B then can be proved by looking at the desired pattern of holes or crossing region of corridors, and by finding near to the place where according to the given pattern there should by a hole or a crossing region of corridors some square-shaped open domain that either entirely belongs or does not belong to Y. Such an open domain may be much smaller than what is expected by the pattern, but one always finds one: either inside one of the holes (each of which must be an open domain), and if some neighbourhood is free of holes this gives an open domain which belongs to Y. Then one uses the mechanism to retract the region as just found to square shape as described in 0.7(2.). Since our pattern is arbitrarily fine and each hole of Y is open, in the course of this process each hole is at a certain state found and is retraced to square shape. That way one can construct a space which is homotopy equivalent to the given one and has all its holes arranged according to some pattern of (approximately, cf. 3.6) vertical and horizontal lines of corridors which do not have the desired boundary coordinates as defined in 0.7(7.), but are arrange according to the same combinatorial pattern. Since the x-coordinates as imposed by the boundaries of the vertical corridors and the y-coordinates that come from the boundaries of the horizontal coordinates each form a dense set, by mapping each of these coordinates to the corresponding desired coordinate of the 0.7(7.)-pattern, the space Y' that was obtained by retracting holes to square shape can be homeomorphically mapped to a space that has a corridor lattice that satisfies all properties of 0.7(7.). Composing this homeomorphism with what has been obtained from retracting all our holes gives the desired homotopy equivalent embedding.

2.5 On the Proof of Theorem C (cf. 1.6):
The map has be characterized in 0.7(6.), and the fact, that it is surjective can be concluded from having constructed its inverse in 2.2.

2.6 On the Proof of Theorem D (cf. 1.7(i)):
The theorem is proven by carefully analyzing the possibilities for non-uniqueness within the reading process. Essentially three facts are proven:
(i) If two word sequences can be read from the same path, they are strongly equivalent (cf. [Zas0].D.2.5).
(ii) If the same word sequence can be read from two different paths, the paths coincide up to monotonic reparametrization (cf. [Zas0].C.4.1(ii)).
(iii) If ω is some word sequence that can be read from a path u and if η is strongly equivalent to ω, then η can also be read from u (cf. [Zas0].D.2.11).
Ad (i): This follows since adjustments of the reading process in the sense of 2.2(i)–(iii) do never need the nesting phenomena as described in 1.8(3.).
Ad (ii): By our convention for word sequences, for each letter that reports an act of crossing a vertical corridor we know the x-coordinates of the start point and of the end point of the crossing segment, and similarly we know the y-coordinates for each letter that is associated with a horizontal corridor. Using this, we can prove for two paths from which the same word sequences can be read to take on the same coordinates, and we can construct a monotonic map that associates the parameter values of the same coordinates with each other.
Ad (iii): The essential argument here is a scheme of proof by contradiction: Look at at some position (cf. 3.7)) where the two word sequences ω and η do not coincide, e.g. where one of them has a cancelling pair AA^{-1} that does not show up in the other word sequence. In such a case we have to show that the path u must perform a tangency so that in the sense on 2.2(i) we are given the freedom to read or not to read these letters from u. Assume that some path realizing η does not perform the tangency, i.e. it keeps a certain distance ε apart from the crucial boundary that is touched by u. On a sufficient higher level the corresponding ε-strip can be filled by many corridors that are parallel to the crucial boundary line, and this would imply that cancellations that could turn ω into η would have to be nested in an unbounded way, similar as described in 1.8(3.). I.e. ω and η could not be strongly equivalent. The same strategy of arguments can be applied for the processes 2.2(ii) and 2.2(iii). Finally the reading adjustments for η can be made compatible by a Königslemma type of argument constructing the "graph of all reading adjustments".

In order to prepare sketching the proofs of Theorems E and F (cf. 1.7(ii), 1.9) we proceed by giving the Definition which of the elementary processes is considered to be of enlarging and which of shortening type:
2.7 Definition of the order between word sequences:
In the second half of this definition we will define for each of the elementary operations between our words to be either of shortening or of enlarging type, so that the inverse of a shortening operation is enlarging and vice versa. This then induces a partial order between words, based on the property whether two words can be transformed into

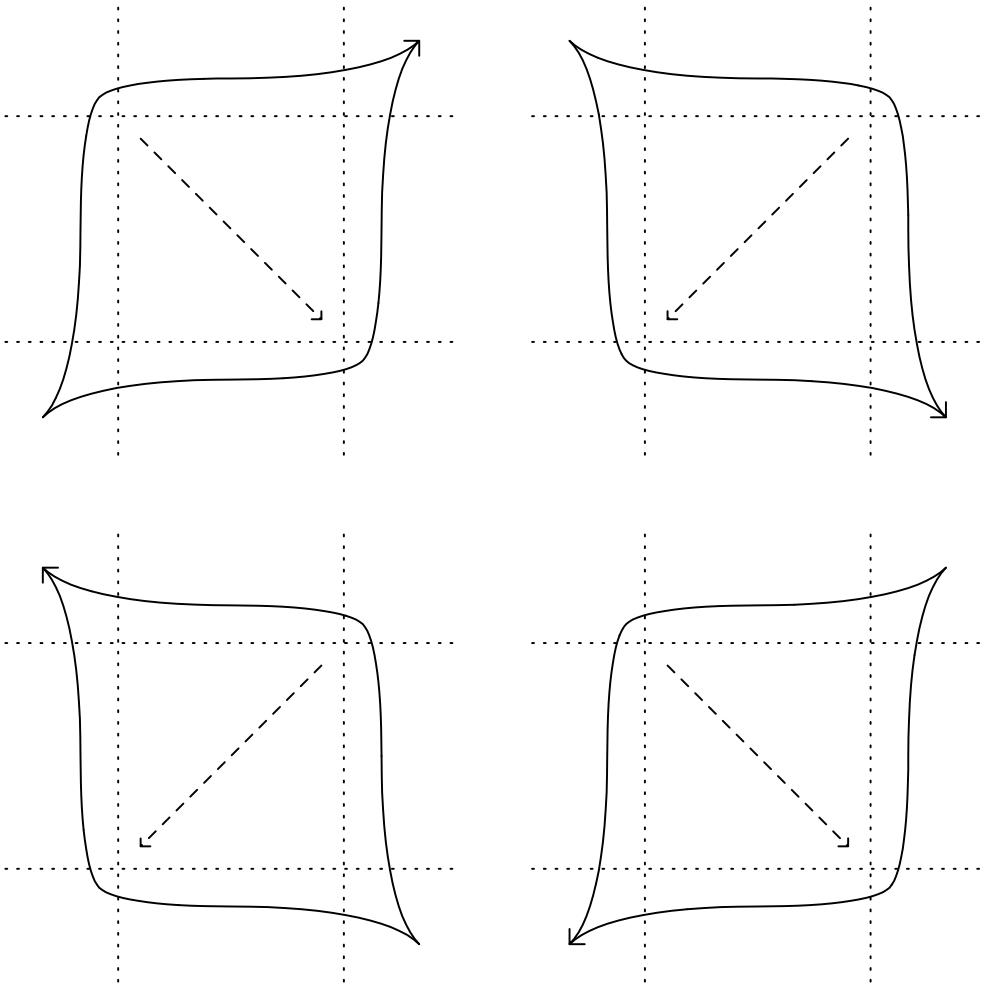

Fig.j (belonging to (iii)):
This figure shows which moves of paths correspond to shortening and which to enlarging transpositions. See (iii) for a more detailed description of how to understand this figure.

each other by using shortening operations only or merely enlarging ones, respectively. Similarly one gets the corresponding definitions for the order between word sequences by requiring that there exists a compatible sequence of transformations between the two word sequences such that all operations occurring on all of the levels are either of shortening or of enlarging type.

Hence, what remains to be done is to classify the elementary operations into shortening or enlarging ones correspondingly:

(i) Cancellations are shortening, insertions are enlarging.

(ii) Truncations are shortening, appendances enlarging.

(iii) The corresponding definition for elementary transpositions is more complicated and symbolically represented in Fig.j. In what follows we describe how to interpret

this figure:

Note that a pair of two adjacent variables, as we have to consider it for our purposes, can essentially be of one of eight types: Namely the first factor could or could not have an exponent "-1", similarly for the second factor, and since for two variables, to be commuting, one of them has to belong to a horizontal corridor and the other to a vertical one, there is an additional choice in which order to put the two variables. For each of these eight types of products we investigate into which type it would be transformed if we apply the corresponding elementary transposition and interchange these two factors. This puts our eight types of products into four pairs of types. Each of these pairs is symbolically represented in one of the subfigures of Fig.j. Each of the subfigures indicates by dotted lines the corridors to be crossed and by solid lines it shows the two alternatives a path could take to cross both corridors. That way each of the subfigures combines the two product types of one pair of types. The arrow between the two path alternatives which has been drawn in dashed mode finally indicates which of the two corresponding product types is to be considered as the shorter one. The arrow points at the alternative to be considered as shorter.

Note that the moral of our definition is that pushing a path either towards the bottom-left corner or towards the bottom-right corner makes a path shorter. As a consequence of this definition, for a right-angeled triangle whose legs of right angle are horizontal or vertical it might be shorter to go over these legs than to go over the hypotenuse.

In contrast to this odd geometrical property we have a fairly simple combinatorial definition, and we have the property that the inverse of a reduced path is again reduced.

2.8 Definition of "far-distance cancellation":

Far-distance cancellation cannot do anything that a sequence of transpositions and cancellations of adjacent variables could not do, also. However, we have to introduce far-distance cancellation as an elementary process which we will be using besides the three ones introduced in 1.8(i)–1.8(iii) equally to overcome the difficulty described below:

There are situations where we can shorten a word by performing a cancellation, but before we get the variables to be cancelled in adjacent positions, we have to do a number of transpositions some of which might be of enlarging type. This phenomenon is a severe obstruction for getting the theory of our equivalence of word sequences as nice as we want to have it (cf. 1.9), i.e. so that irreducible implies minimal. Introducing far-distance cancellations, i.e. allowing to take out two variables which are inverse to each other and are not in adjacent positions but which satisfy that all variables between them do commute with them, is our method of overcoming this obstruction, since we are regarding a far-distance cancellation as an elementary process of its own which is always of shortening type.

The far-distance cancellation has an obvious inverse operation which we, of course, also regard as an elementary legitimate equivalence operation. We call it *"far-distance insertion"*.

2.9 Lemma: *If ω is a word sequence, each of whose words can be derived from the corresponding word of ω by applying a sequence of shortening processes only, then ω' is*

equivalent to ω.

Sketch of Proof: The non-trivial statement of this lemma is that we did not assume our sequence of finite sequences of shortening operations to be compatible (cf. 1.8(2.)), and hence it is the non-trivial task of this proof to construct out of the given information and the internal compatibility of the word sequences ω and ω' such a compatible sequence of finite sequences of shortering operations that can prove the equivalence between ω and ω' in the sense of Definition 1.8:
Compatibility can be obtained by a conclusion essentially using the Königslemma, cf. 2.1. In order to apply this graph-theoretical scheme of conclusion for our purposes, we construct the following tree: The vertices of our tree are given by finite sequences of words $(w_1\,,\,w_2\,,\,\ldots\,,\,w_k)$ that satisfy that for some $i \in \mathbb{N}$ $\;w_1$ is the i^{th} word of $\omega\;$, $\;w_k$ is the i^{th} word of ω' and that w_j results from w_{j-1} for all $j \in \{2, 3, \ldots, n\}$ by applying one elementary shortening operation as defined in 1.8, 2.8 and 2.7 to w_{j-1}. Note that the compatibility properties of a word sequence imply that such a finite sequence of words giving a "reduction scheme" induces a reduction scheme for the $(i-1)^{\text{th}}$ words of our word sequences ω and ω', also. In order to get the induced scheme, one has essentially just to read the $(i-1)^{\text{th}}$ word from any of our words of that finite sequence, and to check elementarily that any of our transformations for i^{th} words on the $(i-1)^{\text{st}}$ level either induces the identity or a similar operation as we had on the i^{th} level. By taking the identity operations out of the corresponding sequence, we get some reduction scheme between the $(i-1)^{\text{st}}$ words of ω and of ω'. We then define that two vertices which are given by induced reduction schemes in the above sentence are to be connected by an edge. Note that by definition any vertex is associated with a level index, and that all edges of our tree are joining vertices with adjacent level indices. Since the reading of the $(i-1)^{\text{st}}$ word not always is unique, there may be more than one edge emanating to the level above. Due to this phenomenon we do not necessarily obtain a tree, but the idea of the Königslemma-conclusion can be applied in spite of that (cf. 2.1(Fig.h_2)). The facts that all vertices and edges of our graph have a clear level structure, that we can run from each vertex monotonicly to the root of the tree and that by assumption there exists at least one vertex on each level of the tree makes it possible to use the same iterative conclusion as in the sketch of the proof of the Königslemma in 2.1 for finding an infinite branch in our graph. Such an infinite branch then gives the desired compatible reduction scheme that proves equivalence between ω and ω'.

In particular note the following special case of the above Lemma 2.9, where ω' is the empty word sequence:

2.10 Corollary: *Let ω be a word sequence each of which words can be transformed to the empty word by shortening elementary operations only. Then ω is equivalent to the empty word sequence.*

2.11 Lemma: *Let ω be as in 2.10. Then ω has a compatible reduction scheme which brings ω to the empty word sequence and which does not contain any elementary transpositions.*

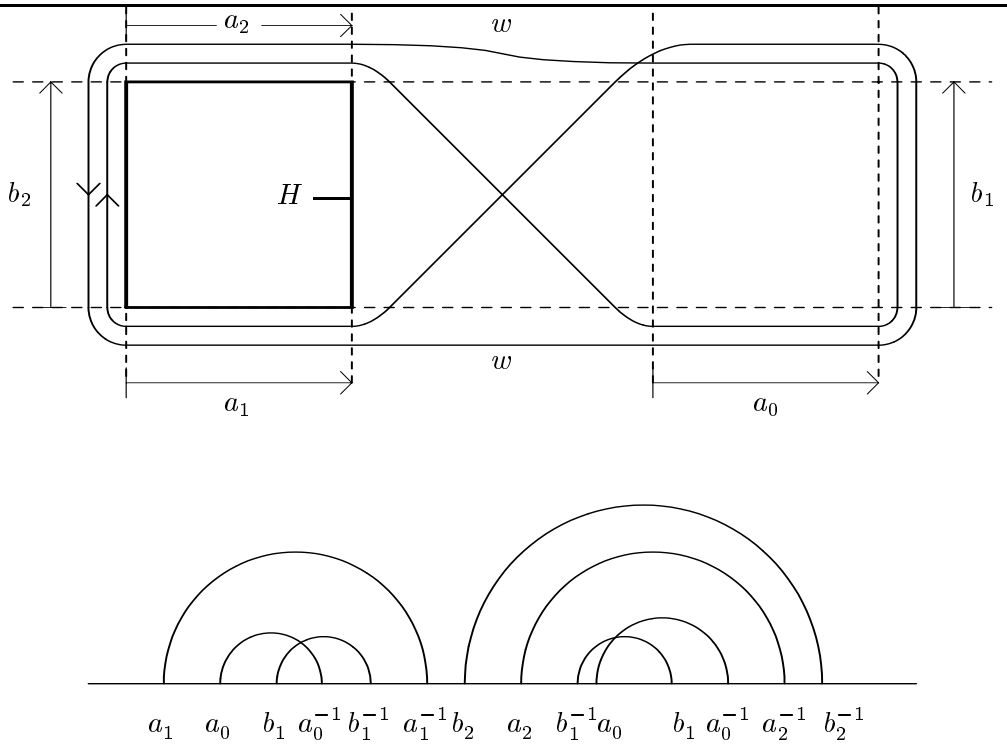

Fig.k (belonging to 2.12):
This figure visualizes a concrete cancellation pattern and shows how in this case, as claimed by Lemma 2.11, transpositions can be avoided.

Sketch of Proof: By assumption there exists a compatible reduction scheme to transform ω into the empty word sequence, but this reduction scheme may contain elementary transpositions also. We then try to transform such a reduction scheme into one which does not have transpositions anymore. In principle we do this, on the one hand, by omiting all transpositions, and on the other hand by performing the same cancellations as in the scheme originally given, even if the corresponding variables now are in different places. However, since we could only perform transpositions if the two variables to be interchanged had been defined to be of commuting type, our omiting of transpositions cannot have made a legitimate cancellation illegitimate, it could only have turned a cancellation of adjacent variables into a far-distance cancellation or vice versa. As in 2.9 we first have to apply these arguments to each word of ω, and then we can use the Königslemma to obtain compatibility for our new reduction scheme, also.

Observe that the following theorem gives Theorem E in the special case of a nullhomotopy (cf. 1.7(ii)):

2.12 Theorem: *ω describes a nullhomotopic closed path, if and only if it satisfies the conditions of Corollary 2.10.*

Sketch of Proof — "$\Longrightarrow$":
Technically this direction is proved by showing the contraposition, i.e. by deducing that

the assumption that at least one word of the word sequences cannot be reduced till it is empty (i.e. that is not "exhaustible") implies that the corresponding path cannot be a closed nullhomotopic one. This is the easy direction of the proof of this theorem, since in this case all computations can be performed in the finitely punctured plane X_i, i.e. in $\mathbb{R}^2$ where only the holes with an index that is smaller than or equal to i are removed. There is one technical lemma to be proven, namely that the exhaustibility of a word that is read from a slice-system of corridors (cf. 0.7(Fig.e)), is equivalent to the exhaustibility of a word that is read from the entire lattice as it is pictured in 0.7(Fig.d_1)–(Fig.d_3). This technical lemma follows since the sliced plane is topologically a disk again and hence does not contain any obstructions for nullhomotopy. Based on this technical lemma the desired proof-direction then follows, since a word that is read from a slice-system of corridors corresponds to the classical possibility of describing the elements of the fundamental group of the finitely punctured plane with respect to a free basis, and hence for such words the non-exhaustibility implies that our path corresponds to a non-trivial element of the fundamental group of a space that is even bigger than our X.

"$\Longleftarrow$"

Note that as a consequence of 2.10/2.9 we get that the word-sequence ω has a compatible reduction scheme. However, in order to construct a null-homotopy out of the combinatorial data of ω, we still need a slightly stronger information, namely that ω has a "compatible cancellation pattern". The difference between these two concepts can be best seen at the following example: Consider the string of letters "$aa^{-1}a$". A reduction scheme only notifies that this string is to be shortened to "a", whereas a cancellation pattern for $aa^{-1}a$ in addition also has to give the information that either the leftmost pair of letters is to be cancelled against each other and that the rightmost letter survives, or vice versa. More generally speaking: A cancellation pattern for a word can be represented by a system of arches each of which connects a factor with another factor represented by the inverse variable so that only those arches may intersect each other which belong to commuting variables (see Fig.k for an example). Such a system of arches as in Fig.k then describes one particular sequences of elementary cancellations that could be carried out in any order represented by the nesting of those arches connecting the non-commuting variables. Since by 2.11 we got rid of transpositions and since, when doing compatible representations for our endpoints (cf. 3.5) we can do without truncations, such a graphical system is able to represent any cancellation pattern in our context†. It is a similar Königslemma-argument as in 2.9 to built up such a compatible cancellation pattern out of the information that there has to exist one for each word. The remainder of the proof is about how to turn such a compatible cancellation pattern into a null-homotopy.

† Hence recall that an arch-system such as Fig.k is just a special case to which, as it seems in this report, we can reduce the situation of nullhomotopy. Fig.l_8 gives the idea of a more general picture and Fig.l_1–Fig.l_7 show the elementary components out of which each of these more general pictures has to be composed.

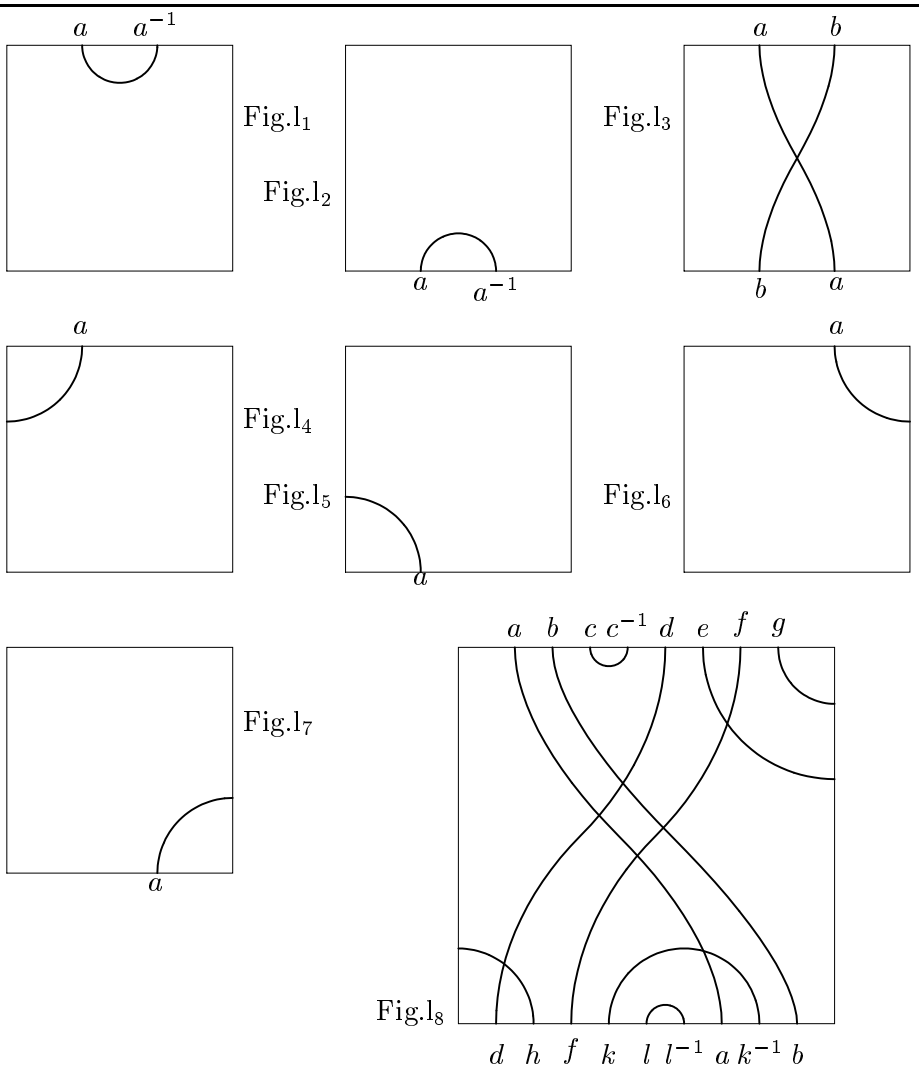

Fig.l (belonging to 2.12("⇐=")):
This figure shows the transformation diagrams for the elementary processes and gives one impression how a combination of these elementary components looks in a concrete example of transforming two words into each other.

Our idea therefore is to perform a similar limit-construction as when in 0.7(6.) we were constructing a path out of combinatorial data. More precisely, our i^{th} map of the sequence approximating our homotopy will be a homotopy that contracts our path in X_i, i.e. in that subset of $\mathbb{R}^2$ which is obtained by removing only those holes with an index smaller than or equal to i. The limit of all those approximating homotopies will be the homotopy for our desired path. The approximating homotopies can now be constructed as follows: Take a round disk D, and put the parametrization of the approximating path on its boundary. Compare this approximation with the i^{th} word of the corresponding word sequence, and mark the beginning and the end of each segment where the path crosses a corridor by associating the segment with the corresponding

letter of our word. Note that the arch system of our cancellation pattern relates each of those segments precisely to another one. Connect the beginning and the end of the corresponding partner-segments with each other by drawing the corresponding lines that are straight with respect to the hyperbolic geometry, assuming that our round disk is metricized as hyperbolic plane according to the scheme of the Poincaré disk model. This defines corridors inside our round disk D. Those corridors can only intersect each other if they belong to commuting variables, i.e. if the corresponding corridors intersect each other in our space of consideration, also. Now extend the definition of our homotopy (which is given on the boundary of D already) by defining it on the boundaries of our corridors as drawn into D. Perform this extension by linearly interpolating the values that are uniquely determined at that points where our corridor boundaries intersect each other and that are already given where our boundaries reach ∂D. We can regard our corridor boundaries as giving a cell-subdivision of D; and by the preceding steps we are already given the values of our homotopies on the one-skeleton of that cell-subdivision. What is left to be done, is to extend these values into the interior of any of those cells. This clearly can be done, since the definition for the boundary for each of those cells only contains values that lie inside the same square, corridor or crossing region of corridors of the lattice of the i^{th} level or in some adjacent of those regions forming a star-shaped region altogether. Hence the definition on this boundary describes a path in a cell without holes, at least without holes that are already contained in the i^{th} corridor lattice. Hence this boundary-path can be considered as nullhomotopic for our purposes, and by plugging these null-homotopies together we extend our definition to all of D obtaining a homotopy for the i^{th} approximating path of our path of consideration.
What is left for doing now is showing that the construction converges and that the limit gives a homotopy for our path of consideration. Note that, since we had been using the parametrization of our original path for all approximations on the boundary of D, we have trivial convergence on the boundary. In addition, it turns out that all what is left for showing is the uniformity of our construction. For, uniformity guarantees the existence of a continuous limit function, which by using the same argument as in 0.7(6.) cannot take on values in any of our holes, hence lies in our space of consideration. Now, uniformity is an immediate consequence from the properties of a compatible cancellation pattern: Note that the behavior of arches as regarded in Fig.k for defining a compatible cancellation pattern immediately translates into the intersection behavior of the corridors drawn into D. Even the non-uniqueness case as pictured in Fig.m gives, that if the values of one our of approximating homotopy of the i^{th} level lie in some of the corridors of the i^{th} level, then on the $(i+1)^{\text{st}}$ level they are in the same area, and if they are in one of the squares of the i^{th} level, then on the $(i+1)^{\text{th}}$ level they are either in the same square or they are in one of the adjacent corridors of the i^{th} level. Based on the this observation the decreasing diameter of our corridor-networks gives the desired uniformity and hence this theorem For more details see [Zas0].■.

2.13 Remark: The moral of 2.12 may be expressed as follows: If a closed path in a subset of the plane is not nullhomotopic, then the obstruction for contracting the path must be visible on a finite level of the corresponding word sequence; and if there is

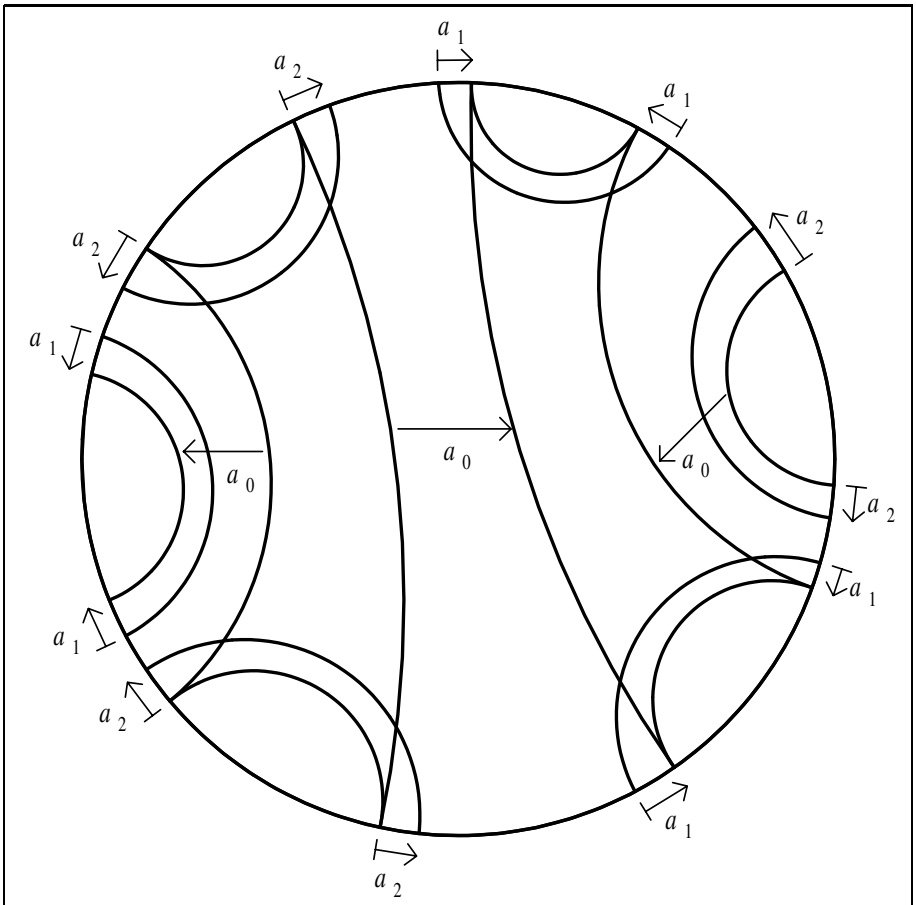

Fig.m$_1$ (belonging to 2.12):
This figure gives an example for a realisation diagram and it pictures two of its levels, cf. Fig.m$_2$.

no such obstruction on a finite level, the associated path must be nullhomotopic. Since according to 0.7(6.) word sequences can be just understood as a sequence of descriptions of approximating paths, this result in particular implies that within the framework given by word sequences there cannot exist non-nullhomotopic paths which are approximated by nullhomotopic ones. Note that the latter sentence gives definitely a false statement in a three-dimensional context:

Example: (inspired by Griffiths, cf. [Gr1].(3.4))

Now consider 0.0(Fig.a) which shows the one-point union of two Hawaiian Earrings, each of which is homotopically trivialized by having a cone attached to it. Define a continuous curve as alternately running over each of the loops of the two Hawaiian Earrings. It is shown in [Zas1].(0.12&3.11), that the resulting curve is not nullhomotopic, even if it is an infinite composite of nullhomotopic loops. In addition note that it is not a contradiction to the idea of word sequences, if we regard those curves which coincide with the given one while running over the first finitely many biggest loops but become constant for the remainder of the parameter interval and hence skip

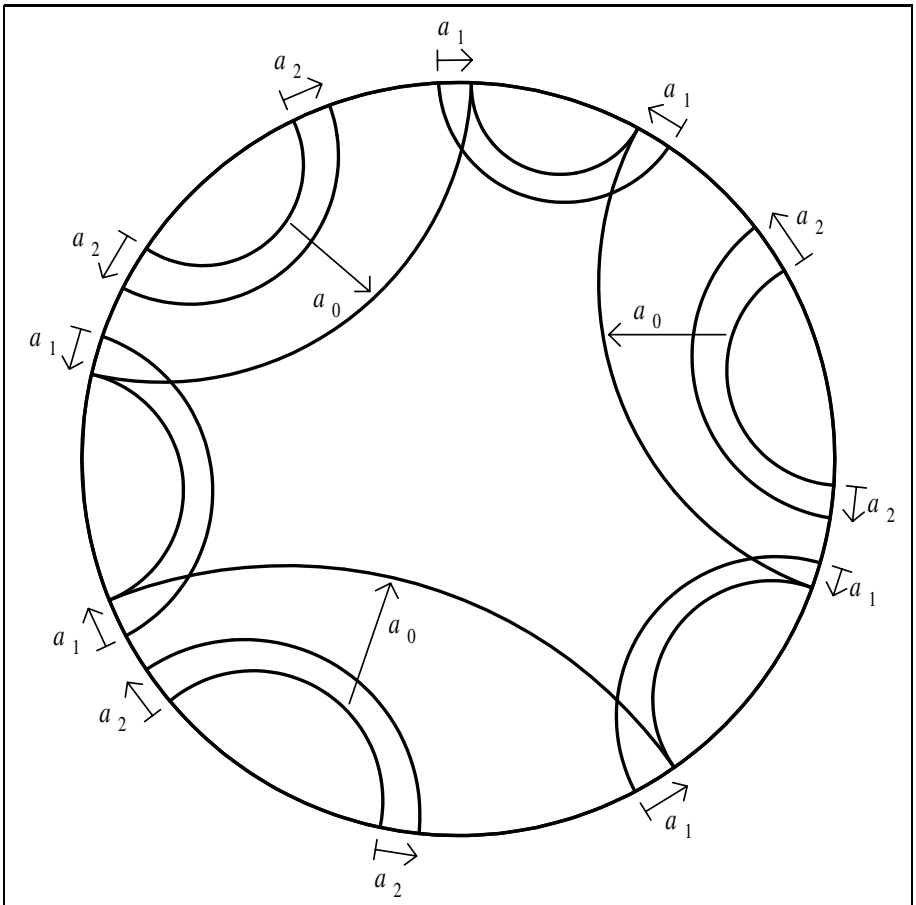

Fig.m$_2$ (belonging to 2.12):
This figure should be viewed in connection with Fig.m$_1$. It provides an alternative way of how a realization diagram might look like that is associated with the same reading that we saw in Fig.m$_1$.

running over the corresponding tiny loops as approximating curves to the given one. And all these curves are, indeed, nullhomotopic.

Remarks:

(i) The concept of describing the obstructions for nullhomotopy by word sequences must be understood as a concept which only works for two dimensions, where, other than in higher dimension, the position of the compressing disk is more or less determined by the position of the curve.

(ii) If two dimensions would permit examples as the above one, our space $\widehat{X_{x_0}}$ could not be metricized by the homotopic distance (cf. 0.4(i)). Namely, in the above example the homotopic distance between the homotopy class as described by our curve and the trivial class would be zero, since there is not any obstruction for contracting any finitely many biggest loops of the described path via the cone points and hence for representing the homotopy class just by the remaining tiny loops. The infimum-

construction of 0.4(i) then gives zero distance, so that in that case the homotopic distance does not give a metric. However, in two dimensions this cannot happen: Here the homotopy distance can be proven to give a valid metric [Zas0].D.12.5), essentially since each obstruction for a nullhomotopy of a path must come from the path going around some of the holes, and each hole has a positive diameter.

2.14 Remark: The purpose of this remark is understanding the origin of the non-uniqueness of reduced word sequences in Theorem F that has made the introduction of firm equivalence necessary (cf. 1.9).

If one attempts to prove the uniqueness of reduced representatives, it appears to be the most natural approach that one firstly tries to look separately at the various levels of a word sequences and to prove for each separate level the uniqueness of the reduced representatives, and then tries to apply the Königslemma-type of arguments for showing that even if one restricts to those operations that can synchronously be extended throughout the word sequence ends up with unique reduced representatives. The first part of this proof-programme can be realized: When restricting to a finite level of a word sequence, the transformations, that appear valid in this level produce a unique reduced representative. However, in the second part of this proof-program, the following phenomenon crops up: Assume that ω and ω' are weakly equivalent word seqences, and that the i^{th} word of ω is denoted by w_1, the i^{th} word of ω' by w_m, and that the tuple $(w_1, w_2, \ldots, w_k, \ldots, w_m)$ describes a sequence of elementary transformations so that $w_j \mapsto w_{j+1}$ is a shortening transformation for $j < k$ and an enlarging transformation for $j \geq k$. Thanks to existence of a unique reduced representative on each level one can arrange the transformations like this. Now, in order to play the Königslemma-argument, we have to try to find a tuple $(v_1, \ldots, v_m)$ so that v_1 and v_m are the $(i-1)^{\text{st}}$ word of ω and of ω', respectively, and so that the transformation $v_j \mapsto v_{j+1}$ corresponds to the transformation $w_j \to w_{j+1}$. However, the synchronization techniques only work smoothly for shortening transformations, and hence $v_2, \ldots, v_k$ have to be obtained by looking how the transformations $w_1 \mapsto w_2 \mapsto w_3 \mapsto \ldots \mapsto w_k$ appear on the $(i-1)^{\text{st}}$ level, while $v_{m-1}, v_{m-2}, \ldots, v_{k+1}, v_k$ are constructed by similarly looking at $w_m \mapsto w_{m-1} \mapsto w_{m-2} \mapsto \ldots \mapsto w_{k+1} \mapsto w_k$. Hence we get two different definitions of v_k, and they need not coincide. It turns out that they coincide up to transpositions of variables, however there is not any warranty that these transpositions (which all can be synchronized by the identity on the i^{th} level) can be arranged in such an order that all shortening transpositions are to be performed before the enlarging ones (cf. [Zas0].D.6.1–2)). Observe that the following example of a path which does not have a unique reduced word sequence is based on precisely that effect:

The path is pictured in 2.20(Fig.p_1) together with the corridors of the i^{th} lattice. We need not assume that there are more holes around than the one that is drawn in that figure; according to 2.7 the position of this path is perfectly short with respect to the legitimate transpositions. Assume that the hole of 2.20(Fig.p_1) belongs to the $(i+2)^{\text{nd}}$ level, then the corresponding $(i+2)^{\text{nd}}$ word is irreducible. The $(i+1)^{\text{st}}$ word seems to be cancellable, since the split-up letters of the corridor B are in a corresponding adjacent position, but we know from the $(i+2)^{\text{nd}}$ level that this cancellation cannot be extended.

On the i^{th} level we either read the word BAB^{-1} or $BB^{-1}A$ or ABB^{-1}, and all these three readings are compatible with the unique reading on the $(i+1)^{\text{st}}$ level. However, the transposition $BAB^{-1} \mapsto ABB^{-1}$ and $BAB^{-1} \mapsto BB^{-1}A$ are both of shortening type, and, since the cancellation of B against B^{-1} is non-exdendible, ABB^{-1} and $BB^{-1}A$ are both possible reduced readings from our path.

The following is essentially a rephrasing of Theorem F (cf. 1.9):

2.15 Theorem: *Within every weak equivalence class of word sequences there exists a firm equivalence class of irreducible representatives, which is reached by shortening every other element of the weak equivalence class.*

Sketch of Proof: In the first part of this proof we consider each word of the word sequence isolately. However, while doing this, we regard only those truncations as legitimate operations that affect a letter that is associated with a corridor that contains the start- or the endpoint of our path. According to the principles as explained in 1.8(3.) other types of truncations could not be compatibly extended throughout our word sequence, and hence for the purposes of our proof this is a legitimate arrangement (cf. [Zas0].D.3.9). As the first step for our proof we try to show for this set-up a similar result as claimed in this theorem. This can be done by elementary combinatorial methods, we suggest one strategy therefore in the following: Invent a "normed cancellation algorithm" (cf. [Zas0].D.4.1)), i.e. make some conventions which of the possible shortening operations has to be applied if more than one is possible. With such a definition it follows from distinguishing enough cases that this algorithm always generates the same word if it is applied to two words which just differ by one elementary operation. This implies that there cannot be more than one reduced word in the equivalence class of an isolated word, since, if two reduced words were equivalent, the normed cancellation algorithm would reduce them to the same reduced word, but, on the other hand, a cancellation algorithm cannot impose any changes to a reduced word. Having proved this result for isolated words, Theorem 2.15 then follows from a Königslemma type of conclusion (cf. 2.1) that we get a similar result when considering word sequences, although the entire uniqueness gets lost (cf. 2.14).

Remark: How the non-uniqueness case as described in 2.14 appears from the technical point of cancelling, is then pictured in Fig.n. In that context in principle two graph structures have to be considered, namely on the one hand we get a graph structure by associating each word with a vertex and connecting those words by edges that can be transformed into each other by one elementary transformation but on the other hand the words from the various index levels have also to be considered and those words of the i^{th} and the $(i+1)^{\text{st}}$ level that satisfy the compatibility demands for belonging to the same word sequence also deserve being related to each other by graph theoretical means.

Now Fig.n visualizes both of these graph structures at the same time: The first of these structures is visualized as a "fat graph", i.e. all vertices are drawn as circles and all edges as corridors, while the second graph structure is visualized by the nesting of vertices and edges, i.e. the circles that correspond to the vertices of fine levels are drawn

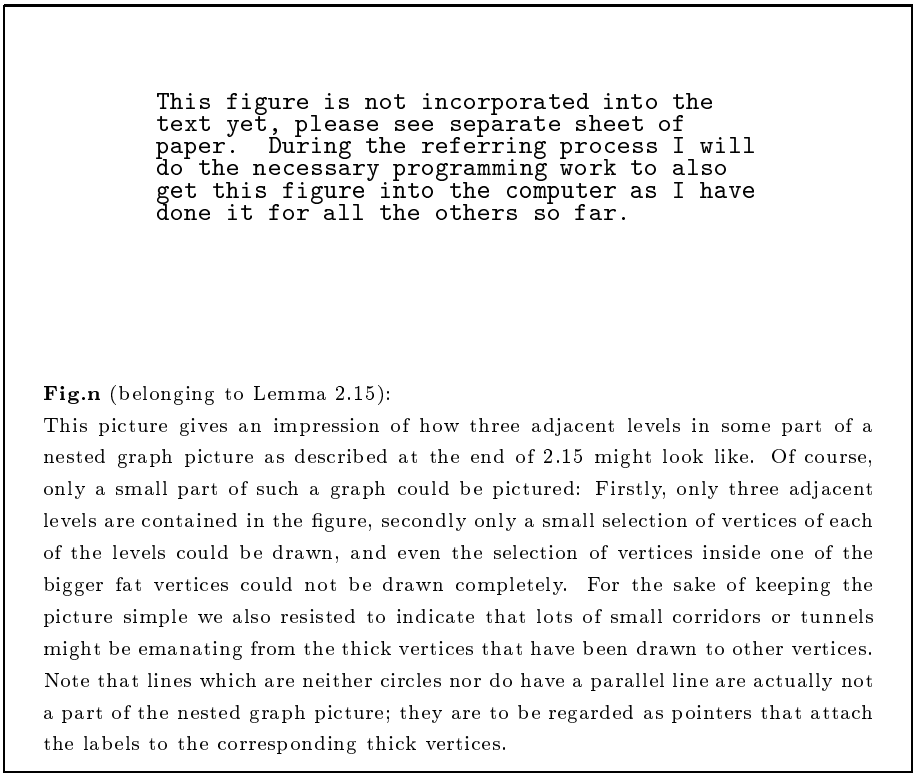

Fig.n (belonging to Lemma 2.15):
This picture gives an impression of how three adjacent levels in some part of a nested graph picture as described at the end of 2.15 might look like. Of course, only a small part of such a graph could be pictured: Firstly, only three adjacent levels are contained in the figure, secondly only a small selection of vertices of each of the levels could be drawn, and even the selection of vertices inside one of the bigger fat vertices could not be drawn completely. For the sake of keeping the picture simple we also resisted to indicate that lots of small corridors or tunnels might be emanating from the thick vertices that have been drawn to other vertices. Note that lines which are neither circles nor do have a parallel line are actually not a part of the nested graph picture; they are to be regarded as pointers that attach the labels to the corresponding thick vertices.

inside corresponding "fat vertices" of the less fine levels to which they are related. This "nested graph" also visualizes whether transformations are of shortening or of enlarging type; the vertices have been carefully placed with respect to the height-coordinate in such a way that any edge that takes one closer to the bottom of a picture corresponds to a shortening elementary process. What appears as "dashed edges" should indeed not be regarded as edges at all, but as "tunnels": Such tunnels connect two vertices if they cannot be joined by a monotonic sequence of edges, but if their associated words just differ by a sequence of transpositions that avoids a complete interchange of two pairs of two neighbouring variables. This is a method of putting firm equivalence into the picture: If a sequence of nested tunnels can be drawn that covers all levels, the corresponding word sequences are firmly equivalent.
The figure seems to contain (since it restricts to three levels only, one cannot tell whether they actually are) two different weak equivalence classes of paths, one of which belongs to the path as pictured in 2.20(Fig.p_1), and the other of which belongs to a similar path that circumvents the hole of 2.20(Fig.p_1) on the other side. For the latter path we do not see an obstruction for cancelling all Bb-letters!

2.16 Corollary: *If two irreducible word sequences are not firmly equivalent, they belong to different weak equivalence classes.*

Proof: An irreducible word sequence represents a local minimum in the graph structure as constructed in the preceding Lemma 2.15; since according to that lemma each graph does not have any local minima apart from the firm equivalent class of global ones, irreducible word sequences that are not firmly equivalent have to belong to different

weak equivalence classes. □

By now we have completed our preparations, and are able to give the

2.17 Outline of the Proof of Theorem E (cf. 1.7(ii)*):*

I.e. we have to sketch how it follows that *two word sequences are weakly equivalent if and only if they represent relatively homotopic paths.*

Sketch of Proof: The proof essentially follows by reducing the situation to the case of nullhomotopies of closed paths that has already been discussed in 2.12:

"$\Longrightarrow$" Let ω and ω' be two word sequences both of which can be reduced to firmly equivalent word sequences ω_0 and ω_0'. By combining these reduction patterns with the natural cancellation of $\omega_0^{-1} \cdot \omega_0'$, we get that $\omega^{-1} \cdot \omega'$ can be reduced to the empty word sequence. Hence by 2.12 the corresponding concatenation of paths is nullhomotopic. Even if the corresponding nullhomotopy need only respect one of the two common endpoints our paths, it can be turned by elementary homotopy theoretical methods into a relative homotopy between the paths associated with ω and ω'.

"$\Longleftarrow$" We try to show this direction by proving the contraposition. So, let us now assume that ω and ω' are two word sequences which can be shortened to not firmly equivalent reduced word sequences ω_0 and ω_0', respectively. We will be done after having proven that $\omega_0^{-1} \cdot \omega_0'$ cannot be equivalent to the empty word sequence, since by 2.12 this implies that the corresponding concatenation of paths is not nullhomotopic and hence that the paths that are associated with ω and ω' cannot be relatively homotopic either. Note that by "$\Longrightarrow$" it is already clear that the path associated with ω is relatively homotopic to the path associated with ω_0, and similar for ω' and ω_0'. So, let us now prove that $\omega_0^{-1} \cdot \omega_0' \not\approx 0$ by assuming the opposite and by showing that this yields a contradiction:

Discuss the possible cancellation patterns, that a word sequence $\omega_0^{-1} \cdot \omega_0$ could have taking into account that ω_0 and ω_0' are reduced word sequences already (cf. [Zas0].D.11.10). Hence there cannot be any cancellations within the letters of ω_0 or within those of ω_0'. Further, we can get rid of truncations by a corresponding specification of our endpoints (cf. 3.5), and, besides this, we can avoid transpositions by 2.11. Thus, all what is left are cancellations of pairs of letters one of which belongs to ω_0 and the other to ω_0'.

If takes some discussing to see that this not only holds for the beginning of a cancellation process, but extends to an entire sequence of shortening transformations. This can be seen by the following indirect argument: Assume that in the course of reducing $\omega_0^{-1} \cdot \omega_0'$ it happens that two letters of ω_0^{-1} can be cancelled against each other, even if they are found in an uncancellable position within the reduced word sequence ω_0^{-1} (a similar argument has to be used if we assume that two such letters can be transposed). In the course of reducing $\omega_0^{-1} \cdot \omega_0'$ it might appear that the terminal segment of ω_0^{-1} is precisely the inverse of ω_0', and they cancel against each other, but of course our two letters cannot have belonged to that segment. After this process has terminated, the next processes can only be transpositions or far-distance cancellations. As stated above, we assume that the performance of such processes removes the obstruction the prevents to cancel the two letters of our consideration against each other inside ω_0^{-1}.

We now consider that level of ω_0^{-1} which within ω_0^{-1} contains the obstruction for cancelling our two letters of consideration against each other and hence in principle we restrict to considering a finitely punctured plane. We also look at the corresponding "thinned word", i.e. at that word that is only read from those corridors which give a slice-system of the correspondingly finitely punctured plane. With similar arguments as in 2.12("$\Rightarrow$") any obstruction for cancelling two letters against each other must be contained in the thinned word, also. Now, the corresponding calculus of letters coming from a slice-system of corridors is precisely the same as for a free basis of free groups. Hence there are not any such things as far-distance-cancellations or transpositions of variables. From such considerations we get a contradiction to our assumption that in the present situation such transpositions and far-distance-cancellations can remove the obstruction of cancelling two letters of ω_0^{-1} against each other that are in an uncancellable position within the reduced word-sequence ω_0^{-1}. This tells us that we end up with no other operations than cancellations of letters of ω_0^{-1} against those ω' for the entire process.

Hence, when thinking of an arch-system as pictured in 2.12(Fig.k), we get a system of arches which are mostly centered around the attaching position of $\omega_0^{-1}\&\omega_0'$, some of the neighbouring arches of which might be intersecting. Such a compatible cancellation pattern can then be turned into a compatible transposition pattern that transforms the word sequence ω_0 into ω_0', implying that ω_0 would be equivalent to ω_0'. However according to 2.16 not firmly equivalent reduced word sequence cannot be equivalent at all. This gives the desired contradiction and hence the theorem.

2.18 On the proof of Theorem G: (cf. 1.10): Already in 0.4(1.)–(2.) we indicated that Thm.G should be proven by a two-step-argument, the first of which steps is the construction of an abstract homotopy combing. The existence of an abstract homotopy combing (that will turn out to be liftable in the sense of 0.4(2.)) is now an immediate corollary from Theorems 2.15 and 2.17: Consider all relative homotopy classes of paths the connect the base point $x_0 = (0,0)$ to an arbitrary point of X, and use that by Theorem 2.17 each of these relative homotopy classes of paths is represented by a weak equivalence class of paths. By Theorem 2.15 this weak equivalence class has a unique firm subclass of reduced word sequences, and, since firm equivalence is stronger than strong equivalence by 1.8, each of the representatives of the firm subclasses represents the same abstract path. Our abstract homotopy combing is then composed out of these paths. Now, lifting this homotopy combing to a continuous parametrized homotopy combing, means essentially, that we have to find some suitable parametrization for each of the combing paths. This parametrization will ensure the desired continuity, if it satisfies that the distance of points with the same parameter values on two different homotopically near paths is bounded by a multiple of their homotopic distance with the multiplicity possibly depending on the average behavior of the two nearby paths. I.e. we will only by able to find a suitable parametrization if we synchronously look at all paths in our homotopy combing, and that is why we call the parametrization to be constructed a *"synchronous parametrization"*. In 2.19/2.20 we will show how to construct a synchronous parametrization, and we will outline, that it actually satisfies

the desired continuity properties. With these items we will complete our sketch of the proof of Theorem G.

2.19 How continuously to parametrize the paths of our homotopy combing? — (cf. [Zas0].§E.2).)

(i) At first we have to be aware that we cannot think of a unique word sequence that is associated with each combing path, but just of a firm equivalence class of such word sequences. With respect to the ambiguity of the reading process as described in 2.2 this implies that

- ▷ tangencies are, wherever possible, disregarded, since this makes word shorter. Under certain circumstances this causes that the segment of some path that is associated with an act of crossing, if due to tangency effects the segment is not clearly defined by the behavior of our path, is not the shortest segment that can be said to perform an act of crossing, but the longest possible segment. In particular, the three consecutive acts of crossing as pictured in 1.8(Fig.g_5) may just be treated as one act.
- ▷ Since truncation are regarded as shortening processes in any case, if ever our paths ends inside a corridor, the corresponding partial act of crossing will not be acknowledged by a letter.
- ▷ If a path crosses two corridors simultaneously by going diagonal through a corridor crossing, despite the specification 2.7 there are some situation where even within a firm equivalence class of reduced word sequences we have to face the arbitrariness in which order to put the two variables (cf. 2.14). Hence we have to make sure that in the following our parametrization formula responds the same to any two firmly equivalent word sequences, even if some of the variables appear in another order. We will achieve this property by essentially treating the x-coordinate and the y-coordinate independent from each other in the sequel.

(ii) We think of our path as given by an arbitrary parametrization $\varphi : [a, b] \to X$. We assume that φ describes our path without constant segments. Let ω be a reduced word sequence that describes φ, and let ω_i be the i^{th} word of it.

(iii) We then define the monotonicly increasing function $\ell_x : [a, b] \to \mathbb{R}_+ \cup \{0\}$ by firstly defining interpolation nodes $a = a_{1,0} \leq a_{1,1} \leq a_{1,2} \leq \ldots \leq a_{1,k_1} = b$ such that each vertical letter of the first word of our word sequence ω gives one of those interpolation nodes — more precisely, such that $\varphi|_{[a_{1,j-1},a_{1,j}]}$ is that segment of φ that starts with the exit of the act of crossing the corridor that is described by the $(j-1)^{\text{st}}$ vertical letter of the first word of our word sequence and ends with the exit of the corresponding j^{th} letter. Hence apart from limit cases we have that $k_1 = \#\{\text{vertical letters of } \omega_1\} + 1$, where $\varphi|_{[a_{1,k_1-1},a_{1,k_1}]}$ is the terminal segment of φ that starts with the exit from the act of crossing the corridor that corresponds to the last vertical letter of ω_1. The concept of defining ℓ_x should be understood that way that in a case where $\varphi|_{[a_{1,j-1},a_{1,j}]}$ is averagely monotonic, we want $\ell_x(a_{1,j}) - \ell_x(a_{1,j-1})$ to be the length of the interval of x-coordinates taken on by $\varphi|_{[a_{1,j-1},a_{1,j}]}$; and in case where $\varphi|_{[a_{1,j-1},a_{1,j}]}$ is not monotonic but exits at the end of this interval into the same direction of x-coordinates from which it had come at the beginning of this segment, then $\ell_x(a_{1,j}) - \ell_x(a_{1,j-1})$ shall be

twice the length of the corresponding interval of x-coordinates. As a general formula that can be understood as an extrapolation of the two special cases as described in the preceding two sentences, we define that $\ell_x(a_{1,j}) - \ell_x(a_{1,j-1})$ results from taking twice the length of the interval of x-coordinates that is taken on by $\varphi_{|[a_{1,j-1},a_{1,j}]}$, and by deducting from this length the absolute value of the difference between the x-coordinates of $\varphi(a_{1,j})$ and of $\varphi(a_{1,j-1})$. This defines ℓ_x on our set of interpolation nodes $\{a_{1,0},\ldots,a_{1,k_1}\}$, if we, in addition, give that $\ell_x(a_{1,0}) = 0$.

(iv) Similarly we define some values of a function ℓ_y by using an entire different set of interpolation nodes $a = b_{1,0} \leq b_{1,1} \leq b_{1,2} \leq \ldots \leq b_{1,k'_1} = b$ based on considering the crossings of the horizontal corridors only.

(v) The extensions of the definitions of ℓ_x and ℓ_y from these initial nodes follow in principle by similarly considering the corridor-crossings of the finer lattices, so that at the end we get definitions for a dense set of interpolations nodes and obtain functions $\ell_x : [a,b] \to [0,?]$ and $\ell_y : [a,b] \to [0,?]$ by continuous extensions. We will refer to these functions as "x-length" and as "y-length". Note that when iterating the above two steps we cannot always achieve that $\varphi(a_{i,j}) - \varphi(a_{i,j-1})$ is the (eventually doubled) length of the interval of x-coordinates covered by $\varphi_{|[a_{i,j-1},a_{i,j}]}$ that has been used in Step (iii) of this definition, since this definition would not respect the values as fixed in the preceding $(i-1)^{\text{st}}$ step. Hence for $i \geq 2$ our definition is that $\varphi(a_{i,j}) - \varphi(a_{i,j-1})$ is obtained by dividing the (eventually doubled) length of the corresponding interval of x-coordinates by s where s is adjusted for each "interval of interpolation nodes", so that the definition of the preceding steps can be matched. Here an "interval of interpolation nodes" is to be understood as $[a_{i-1,k-1}, a_{i-1,k}]$, i.e. it is bounded by two interpolation nodes of the preceding level that are adjacent on that level. Hence the same s-value is adjusted for a number of adjacent intervals of the present i^{th} level that cover one interval $[a_{i-1,k-1}, a_{i-1,k}]$; note that we have $a_{i-1,k-1} = a_{i,j} \leq a_{i,j+1} \leq \ldots \leq a_{i,j'} = a_{i-1,k}$ for appropriate j and j'. Similarly ℓ_y is defined by using an independent network of $b_{?,?}$-interpolation nodes.

(vi) Conclude that the above definition gives that ℓ_x and ℓ_y are monotonic functions which can only take on constant segments if our path has an entirely vertical or horizontal segment.

(vii) Conclude that the "total length" $\ell := \ell_x + \ell_y$ is a strictly monotonic function and use $\varphi \circ \ell^{-1}$ as parametrization for the path of our homotopy combing. Prove that this function $\varphi \circ \ell^{-1}$ does only depend on the trace of the path φ, but not on its arbitrary parametrization.

2.20 Theorem: *The parametrized homotopy combing that is constructed by using the synchronous parametrizations for the abstract homotopy combing as constructed 2.19 defines a continuous function* $\widehat{X_{x_0}} \to \widetilde{\mathcal{L}_{x_0}}(X)$ (cf. 1.10/0.4(1.)–(2.)).

Sketch of Proof: Basically, we want to prove a classical ε-η-estimate for continuity. Let $\varepsilon > 0$ be given and consider some point in our preimage $\widehat{X_{x_0}}$, and note that such a point is represented by a relative homotopy class $[u]$ where u is a combing path that connects the basepoint x_0 to some point $x \in X$. We then want to find a homotopic

distance η that suffices to make sure that all combing paths of homotopy classes α that are not further apart than η from $[u]$, are never further apart than ε for any of their parameter values. In order to find the corresponding η proceed as follows: First consider the $\varepsilon/17$-neighbourhood of x and then find inside of that neighbourhood a smaller neighbourhood which is compatible with the corridor-network of a sufficiently high index i. This means: our smaller neighbourhood should consist of a (5×5)-matrix of squares as in 0.7(Fig.d_1)–(Fig.d_3), where the outside squares of that region all are corridors of the corresponding direction (vertical or horizontal), if they are not holes. With that set-up it can be told from the corresponding i^{th} word of our word sequence whether our path leaves our region again if it has once gone deep enough into it. Inside this lattice-compatible neighbourhood of x we choose a square shaped smaller neighbourhood, which in the sequel of this proof we will refer to as the "η-square". We are postponing to explain how to choose the edge-length η of that square. We then choose a point x' in the η-square and a combing path u' that connects x_0 to x' and is homotopically η-near to u. To wit $[u']$ is considered as a point in the η-neighbourhood of $[u]$ in $\widehat{X_{x_0}}$. Since $[u]$ and $[u']$ are η-close, there exits a path v running from x to x' that remains inside the η-square. We may choose v in such a way that its associated word sequence is reduced. Note that the reduction process of word sequences may shift the associated paths out of round neighbourhoods, but not of square-shaped ones. Let $\omega, \omega', \omega''$ be the word sequences that are associated with u, u' and v. For ω and ω'' we use a compatible philosopy (cf. 3.5) whether or not to acknowledge a partial act of crossing so that $u \cdot v$ can be directly described by the concatenation of word sequences. By Theorem 2.15 and by using the fact that ω' is only defined up to firm homotopy equivalence, it follows that ω' results from reducing the word sequence $\omega\omega''$. The next big step rests in showing that from what can happen during this cancellation process one gets that our two paths, when approximating the η-square, essentially have to go through the following three phases (cf. [Zas0].E.4.13)):

Phase 1. The two paths can do any movements, but they coincide.

Phase 2. After reaching their "falling-apart point" at the end of Phase 1, both paths behave monotonicly in both coordinates, and in one of them they must even be monotonic in the same direction. At the end of Phase 2 both paths take on values on the boundary of or inside of the η-square.

Phase 3. In this phase our paths are not bound by monotonicity conditions any more; they can do any movements as long as they stay inside the η-square.

The non-trivial statement of this phase-classification rests in the fact that, if the falling-apart point lies outside the η-square, the paths are after that point and before entering the η-square bound by the above monotonicity conditions. Of course, there need not be such a monotonicity phase: Phase 2 could be empty, if the falling apart point lies inside the η-square. In order to show that, if the latter property is not given, we have to have the monotonicity conditions of Phase 2, we need to analyze the possible transpositions or far-distance cancellations that could take place when performing cancellations in the concatenation of our two reduced word sequences in order to describe the extension of our path between x_0 and x to a path connecting x_0 with x': The result is mainly a

consequence of the fact that only horizontal and vertical variables can commute with each other (cf. 1.8(ii)), and that, if one of our word sequences should have a non-monotonic behavior (i.e. has on some of the levels variables which occur with positive and with negative exponents), then this word sequence has to contain an obstruction for performing certain transpositions or far-distance operations, because otherwise our word sequence could not have been reduced. On the other hand, it takes precisely such far-distance operations if the falling-apart point, (which is x when the paths as originally given are concatenated), should be moved outside the η-square, as a consequence of transforming the concatenated word sequence of the reduced paths into a reduced one again.

In fact, having the monotonicity of Phase 2 is an important argument for estimating that the two paths can never fall further apart then a fixed multiple of η. More precisely, we have to estimate the maximal distance between two points with the same parameter value, and we have to note that even before the falling apart of the two traces there may be points with the same parameter value which do not coincide. This can already happen as soon as our two paths have entered that square of the first lattice network that contains the falling apart point of their traces. However, by using the monotonicity of Phase 2 the difference of two points with the same parameter value can essentially be estimated by a multiple of η, but this estimate requires a careful analysis of the phenomena of the parametrization of our paths as established in 2.19. In particular we need to take a close look at at the corresponding s-factors as used in 2.19(v) and we will have to distinguish the possible cases at which places in our three-phase-picture the boundaries of our corridor lattices could lie, one particular nasty case of which is pictured in Fig.p_2. However, the corresponding careful analysis gives, that the average speed (i.e. the speed which is only measured with respect to the corridor exit points of a certain fixed corridor lattice) of our two combing paths is very similar in Phases 1 and 2, the coinciding behaviour in Phase 1 and the monotonic behavior in Phase 2 being the most important argument for the corresponding estimate. Hence we can conclude that two points of our path with the same parameter value can never be further apart than a certain multiple of η, where the above "average speed" is the essential factor of the corresponding multiplicity. This suffices to prove the (not necessarily uniform) continuity of the contraction process. Indeed, the entire homotopy combing must be considered as a construction for which a property like uniform continuity cannot be expected; of course that part of the homotopy combing which is actually used for contracting our sphere S^k will satisfy a corresponding uniformity property by using standard compactness arguments.

2.21 On the Proof of Theorem A (cf. the beginning of this paper):
Given a map $f : S^k \to Y$. Since we will succeed in finding a contraction for f within its image domain $f(S^k)$, the situation reduces to compact, path-connected and locally path-connected spaces. Since homotopy groups are invariant with respect to homotopy equivalence, our Embedding Theorem 1.2 then reduces the situation to those spaces X which are obtained from the modified Sierpinski Carpet by filling up some of its holes. These spaces permit the construction of a corridor network and hence by 2.19–2.20 the

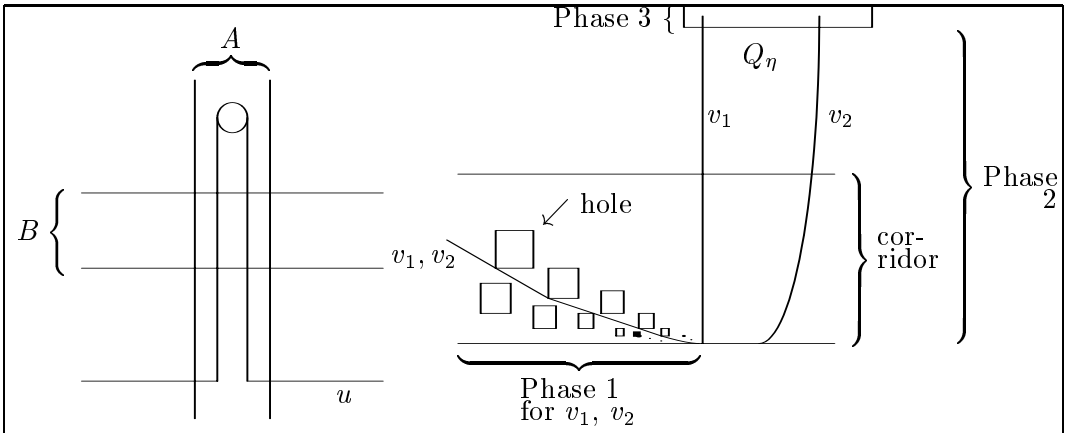

Fig.p (belonging to 2.20):

Fig.p$_1$ shows a path which cannot be represented by a unique reduced word sequence. This situation has also been discussed in 2.14 already.

Fig.p$_2$ showns the falling-apart of two paths as according to our construction in 2.20 it happens at the end of Phase 1, and a particular nasty situation for where a corridor-boundary could lie with respect to the falling-apart point. In particular observe that a boundary-behavior like in Fig.p$_2$ could in some situations be combined with the non-uniqueness of the sequence of letters as we see it in Fig.p$_1$.

construction of a continuous parametrized homotopy combing. It was already explained in 0.4, why the existence of such a homotopy combing implies that any such space is aspherical.

§3. Some further aspects of the proofs that have not been mentioned so far

3.1 Remark: The realization process of word sequences is technically not that deep, but it is also not quite as smooth as we indicated it in 0.7(6.). The necessary uniform convergence of approximating paths against some limit path has to be achieved on the level of parametrized paths. The parametrization can be arbitrarily chosen in the course of the construction, but it has to be made sure that the definitions are of that kind that they avoid degeneration in all cases. Some effects that might happen are pictured in Fig.q. In this picture each intersection point of a path with a corridor boundary has been annotated by some fraction $\in [0,1]$, and the intention of each fraction is that it gives the parameter value at which the approximating path is supposed to intersect the corresponding corridor boundary. In the course of the iterative process of parametrizing such paths, fixing such values must be understood as defining interpolation nodes for the parametrization. To avoid further use of variables the parameter values have for the i^{th} level been marked by using the assumption that they are all equidistant. This assumption is not realistic, since the speed-up phenomena (cf. 0.6) of the preceding levels will in most situations have caused some distortion. However, our unrealistic annotation

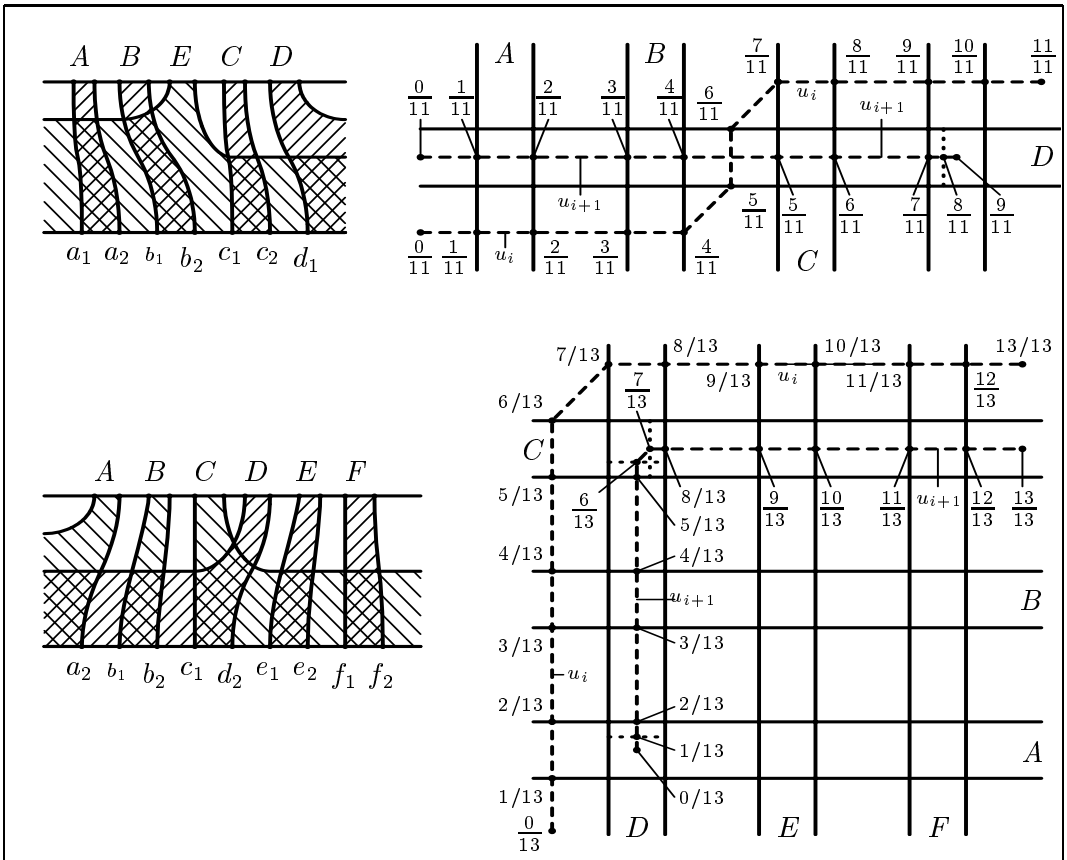

Fig.q1 (belonging to 3.1):
This figure contains four split-up diagrams that show either the entire or just initial subsegments of the i^{th} and the $(i+1)^{\text{st}}$ word of a word sequence. For each of these split-up diagrams we also give up to trivial symmetries and reversion effects the pictures of the corresponding approximating paths. For that purpose the corridors of the i^{th} level have been drawn by solid lines, the corresponding approximating paths in dashed lines. We did not go for the entire symmetric picture of Fig.d, i.e. we have drawn the width of the corridors smaller than the width of the gap-lines so that one gets a clear picture of what are the corridors and what are the gap-lines. Corridor boundaries of the $(i+1)^{\text{st}}$ level are only partially put into this picture, where they are needed to hold the segmentation points of somehow exceptional letters. Where they are pictured, they are drawn by dotted lines.

eases our pictures in such a way, that we just need one more variable for the parameter values of the nodes, namely "k" which we use for the total number of node-intervals in those subpictures, where we only see the initial segment.

Despite of the naturality of the construction, observe that our figures contain situations, where the same interpolation node of the approximating path for the $(i+1)^{\text{st}}$ level is with respect to the i^{th} lattice found two boundary lines apart from the place where it occurred for the i^{th} approximating path. However, since essentially all cases are covered by these figures, this "two" gives a factor which can be used as a global bound in our estimates and does not obstruct uniformity.

3.2 On the unavoidibility of the non-uniqueness of Ψ:

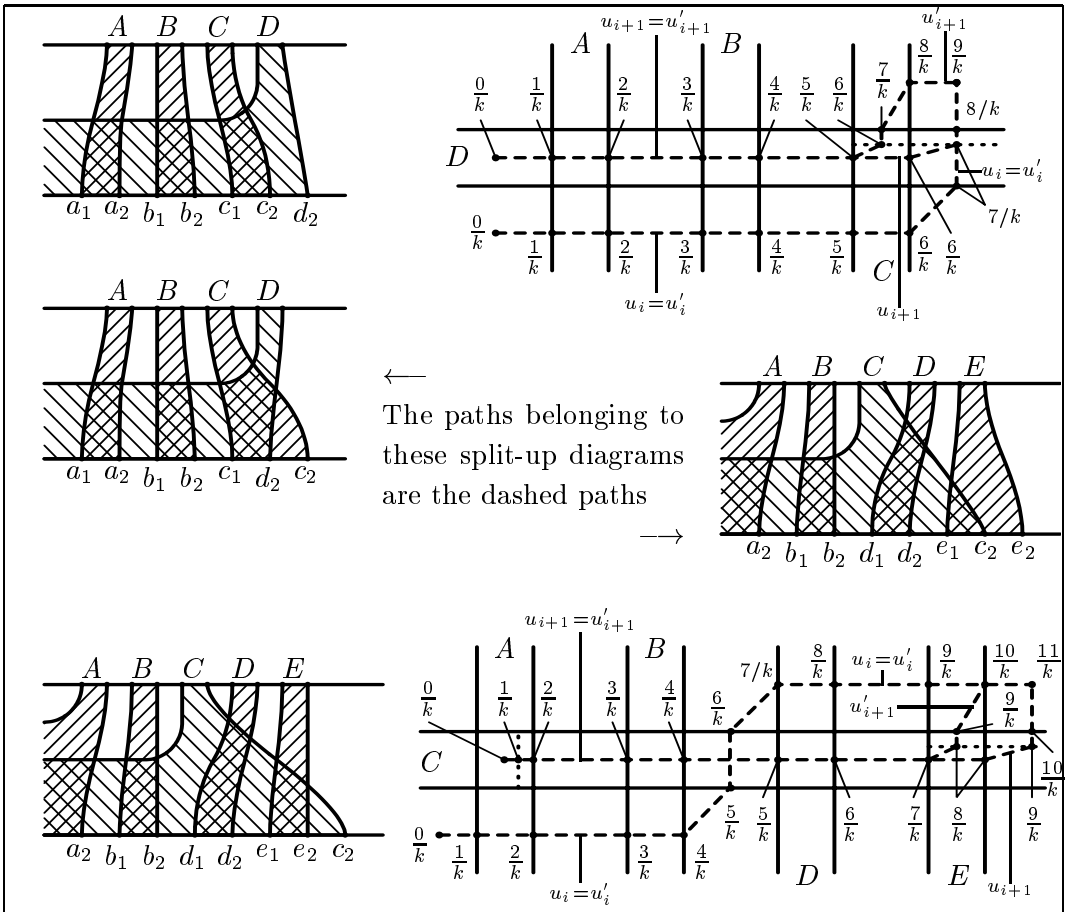

Fig.q2 (belonging to 3.1):
With the same convention as for Fig.q1 this figure gives four more split-up diagrams and the corresponding paths. This time two paths are drawn in each of the corresponding subfigures simultaneously.

The origins of this non-uniqueness can be best understood if one regards of word sequences as a two-dimensional analogue of a theory of describing the real numbers by sequences of digits. In both cases we have a continuous space to be described by infinite sequences of finite strings of combinatorial data; for word sequences this is the variety of paths in one of our spaces X. Hence the infinite continuous space has somehow to be subdivided into "boxes" which are defined as those regions where the first finitely many entries of our combinatorial data are the same. When viewed from this point, it is obvious that it hardly can be avoided that there will exist boxes with common boundaries and that points lying on these boundaries can be approximated from all of the adjacent boxes (cf. 0.7(ii)). Such phenomena are well-known from the reals, e.g. we have $1 = 0.9999\ldots$. Of course, we have similar but more manifold phenomena in the context of this paper, the list of geometric origins of which was given in 2.2(i)–(iii). As for the reals one could make up definitions that associate with each element of the continuous space one of the combinatorial objects only, but such definitions would

hardly be consistent with all of the operations one could perform inside the space; and so one has to be aware that such operations, when performed on the combinatorial level, always could generate one of the aliases of our element of the continuous space. E.g., when computing $0.3333\ldots$ by 3 by manipulating digit sequences, and expecting "1" as the preferred digit sequence for the corresponding real, could you think of a sensible algorithm which does not output $0.9999\ldots$?

3.3 Remarks on the constant path (cf. [Zas0].D.12.2)):
To give some idea what limit cases in the course of the discussion 2.3 need to be considered in [Zas0], in the following we want to discuss how many letters may occur in the word sequence describing a constant path.
Recall that by 2.2(iii) an incompleted act of crossing some corridor may be acknowledged by a letter. To be compatible with the concatenation of paths, we hence must also allow to treat a path which has started in the interior of a corridor and ended in the same corridor as a path which has performed an incompleted act of crossing, and thus put the corresponding letter into the word sequence. We even must allow this, if the direction in which the path actually runs is precisely the opposite direction that seems to be indicated by the corresponding letter of the word sequence (as long as the path does not leave the corridor). Consequently, a constant path which takes on its value in the interior of a corridor may be treated as a path which performs an incompleted act of crossing, and hence we may find some letter in its word sequence. The situation is even worse, if the constant path takes on its value on the boundary line of some corridor. To be consistent with our definition, we then must allow to treat this path as some path, which in the interior of the corridor has started an act of crossing, has completed this act of crossing by reaching the boundary, has then turned around and then started a new act of crossing in the reversed orientation which, even though it remained incompleted, is again to be acknowledged by a letter. Observe that a constant path can be a limit of such a path! – Hence the latter interpretation may already bring two letters on one level into a word sequence of a constant path. Finally, if a constant path has its value on the crossing point of the boundary lines of a horizontal and a vertical corridor, then from each of these corridors two letters may come into the word of a word sequence, and this makes four letters which thus a word sequence of a constant path may have on each of its levels.
However, this "four" is the utmost possibility. I.e. in the combinatorial analysis [Zas0].C.4.8(iii) of word sequences we can argue that if five letters show up in one word, then one of them must describe an actual completed act of crossing, and so with the argument of 2.6(Ad iii) the number of letters in our word sequence grows at least with the powers of two. Conversely, bounded word sequences only belong to constant paths, and they may only have up to four letters on each of their levels.

3.4 On the concept of Triple Sequences:
Consider a path which is entirely horizontal. Consequently, all letters that can be found in the associated word sequence of such a path come from vertical corridors. Conversely, having only "vertical letters" in our word sequence makes it clear that the path is entirely horizontal, but if the intersection of the interval of y-coordinates of all crossed corridors

should consist of more than one point (and this can perfectly happen), it is not possible to reconstruct a path from the given word sequence. This is one of the "unusual" effects that has been neglected in our sketch of the construction of the realization mapping in 0.7(6.) (cf. also 1.5(i)). However, these effects in principle force us to replace the concept of word sequences by the more sophisticated concept of "triple sequences". In a triple sequence the middle entry of each triple is the word as defined 0.7(4.), the first entry is the lattice square containing the initial point of our path and the last entry also is a square of our lattice, this time for describing the endpoint of our path. Each square is to be understood as containing all of its adjacent corridors, hence we call such squares *"extended lattice squares"* in contrast to the "pure lattice squares" that already have been introduced in 0.7(7.). Of course, the compatibility properties as described in 1.5(ii) and 1.8(2.) have to be extended to the situation of triple sequences, in particular the appropriate compatibility between the first or last entry of our triple, and the first and last letter of the word that is the middle entry of the corresponding triple has to be a part of the combinatorial definition of triple sequences. For triple sequences truncation phenomena (cf. 1.8(iii)) then become processes where the description of the start- or the endpoint of the path by the square nesting of the sequences of the first or last entries is adjusted (this can be done if a point is lying in an overlap region), and the middle entries have accordingly to be adjusted, too.

However, although in principle triple sequences provide the only well-defined combinatorial concept for our purposes, even in [Zas0] most of the arguments are just presented for word sequences. However, word sequences are merely used as abbreviations for triple sequences. Observe that in a "usual" situation, i.e. where each word has horizontal and vertical letters, for each level the start-square and the end-square that are compatible with the given word can be deduced from it. I.e. the first and the last entry of a triple sequence are normally determined by the middle entry. The cases where only the first finitely many entries of a word sequence do not satisfy this condition, are not of that huge importance. And the situation where they all do not satisfy the condition is the situation of an entirely horizontal or vertical path. By understanding that the constant coordinate of such a path is determined by the combinatorics (even though it cannot be deduced from the given word sequence), these special cases may be regarded as included into our general discussion as it is presented for word sequences. Observe that for a homotopy combing as it is constructed in 2.19–2.20 the possible non-uniqueness of horizontal or vertical paths is irrelevant either, since in that case the constant coordinate must be that of the start point (0,0).

Another aspect of triple sequences should be mentioned (cf. [Zas0].D.7.17.):

An attempt to prove a version of Theorem 1.9 for triple sequences will result in finding that irreducible triple sequences, unlike as it has been by Theorem 1.9 stated for word sequences, can differ by more phenomena then just by those of firm equivalence. The geometric origin of these additional phenomena can be best visualized when considering a path that is entirely horizontal, and that runs in the interior of a horizontal corridor parallel to its boundaries. According to 3.3 such a pattern could be said to incompletely cross this corridor in the vertical direction, but, of course, such a letter will not appear

in a minimal word sequence. However, even though the word sequence does not report such an act of crossing, the triple sequence has to enregister the start point and the endpoint of the path on one of the sides of our corridor. And switching between these two possibilities of enregistration is not an act of firm equivalence. It is rather an act of firstly appending a (horizontal) letter which reports an act of (partially) crossing our corridor in the vertical direction and then to truncate the letter to the other end of the word again. Since the word can only consist of vertical letters and since the path actually lies in the interior of our corridor, there must be enough commutativity around to perform this truncation as a far-distance truncation. However, since the appendance was not a shortening transformation, this highlights a further obstruction to having unique reduced representatives of triple sequences. However, it can be proved to being essentially the only effect that occurs in addition to those of word sequences. But this effect is not restricted to entirely horizontal and entirely vertical paths, it can happen to all path that on some low-indexed level remain entirely in the same corridor.

3.5 On free & fixed readings and free & fixed equivalences:
The concept of triple sequences as it has been explained in 3.4 should have again highlighted that for a path which starts inside a corridor it is an arbitrary decision of the reading process whether or not to treat the corresponding initial segment inside the corridor as a partial act of crossing and to acknowledge it by putting the corresponding letter into the word sequence. Hence in a full treatment there need to exist *"free reading processes"* where these decisions are a part of the ambiguity of the reading process, and *"fixed reading processes"* where (this can now be expressed in the language of triple sequences), the first and the last entries of a triple are given and the reading is performed with the idea to find compatible middle entries for our triples. If the first and the last entries are valid, there is never an obstruction for the existence of such middle entries ([Zas0].D.2.2(i)–(ii)). These fixed reading processes are of importance when it comes to read various sections of a path separately, so that the word sequences as read from the segments give by mere concatenation the word sequence of the entire path.
Similarly it is sometime necessary to restrict our equivalence relations to those processes which respect the first and the last entry of the corresponding triple sequence. This is precisely done by forbidding truncations and appendances that are performed either to the beginning or to the end of a word. Hence actually there are not just three equivalence relations in [Zas0] (cf. 1.8), but nine ones (cf. [Zas0].D.3.6), since by combining then "beginning"- and the "end"-option four different equivalence relations can be made out of our weak one, and four out of our strong one. However, the firm equivalence relation remains unaffected by this, since here truncations have not been a permitted operation at all.

3.6 Some aspects of the construction of the space Y' within the proof of Theorem B in 2.4:
Recall that in 2.4 in the course of proving our Theorem B an intermediate state has been constructed as a topological space Y', which may be regarded as space with a *"distorted corridor network"*. The corridor network of Y' was said to have the same combinatorial pattern as the corridor networks of those space which are denoted by X in this report,

but with other widths of holes and corridors. The purpose of this paragraph is to give the arguments that one needs to make sure that the holes of Y' that are all retracts of the holes of an arbitrary compact, path-connected and locally path-connected space Y, if according to the pattern 0.7(7.) these boundary lines should lie on one vertical or horizontal line, can indeed all be retracted on such a way that the boundary lines of the retracts satisfy this condition:

First observe that apart from enforcing such a condition by construction, Theorem B might also be proved by another strategy, namely the strategy that the corridors of Y' that have the combinatorial attributes "horizontal" or "vertical", do not have boundary lines that are precisely horizontal or vertical: If they just differ a bit from the ideal horizontal or vertical position, and if "a bit" is made precise by appropriate demands on the construction, making sure that degeneration effects within limit processes within the construction of the network can be ruled out, this should suffice to map Y' and its distorted corridor lattice homeomorphically to our space X with its perfect 0.7(7.)-corridor lattice, and hence to follow the original scheme of the proof of Theorem B. However, it appears that the simplest method for proving Theorem B is actually to make sure that all boundary lines of the Y'-lattice are indeed entirely horizontal or vertical, and the remainder of this paragraph is for describing the most important arguments for such a Y'-construction:

(i) *What to do to align the boundaries of the holes of one lattice appropriately?* — Consider the squares simultaneously in which according to the given pattern holes and crossing regions of a given lattice are to be found. For each of these squares consider the subset that corresponds to $\overset{\circ}{Y} \cup C$, where C is the complement of Y. Since Y is closed, this set must be open and dense. Now identify all these square-regions by the appropriate translations. This identification gives a finite number of open and dense sets inside one square. By Baire's result, the intersection of these sets are still dense, i.e. it is not empty, and as a finite intersection it in particular is open. Consequently, there exist some subsquare inside this open intersection. Undoing our identification process brings the latter subsquare back into all those square regions, that had been considered as a start point of this construction (i). All holes can be retracted to those subsquares, and by construction they have appropriately aligned boundary lines.

(ii) *What to do to align the boundaries of the holes of different lattices appropriately?* —

Observe that the retraction processes of 0.7(2.) cannot guarantee that, where some boundary according to the 0.7(7.)-pattern should lie on a line which is induced by holes that are created on a prior step, we will find an open domain to which the finer holes can be retracted so a to meet this condition. For that purpose we need to create a *"redrawing process"*. "Redrawing" means that if with the tools as explained above a space Y_i' is created that has a corridor lattice satisfying all purely combinatorial demands of 0.7(2) up to the level index i, then the boundary lines of the corridors of Y_i' will not necessarily be also used for Y_{i+1}'. Instead of this, when passing from Y_i' to Y_{i+1}' for Y_{i+1}' the old the boundary lines may have been moved to the interior of the Y_i'-corridors. That way one has some facility to make sure that the boundary lines of

the new Y'_{i+1}-holes can be aligned with the boundary lines of holes that come from lattices with smaller indices — these holes are then just made a bit smaller by a further retraction process. The boundary lines of Y' then result as limit positions of this iterated redrawing processes. It can be ruled out that this limit causes a degenerated result for Y': Between each two boundary lines that by the redrawing process may be moved towards each other, there are on the next finer level boundary lines which by redrawing processes can only be moved into the opposite direction. Since by the given arguments each finite approximation step can be constructed without degeneration, the additional consideration of the second two boundary lines makes it clear that the first two boundary lines cannot clash in the limit situation.

3.7 On the combinatorial concept of position sequences:
The purpose of the last section of this report is to mention another technical aspect, whose discussion requires some space in [Zas0], but which has been completely neglected in all discussions of this report, so far. This is the effort that it takes to precisely state what are the "positions" within a word sequence (cf. 2.6(Ad iii)). Position sequences are the combinatorial objects that are invented to precisely define such positions within a word sequence. Note that even the simpler concept of word sequences as developed for the Hawaiian Earrings in [Zas2] already needed position sequences: As for the [Zas2]-position sequences, our position sequences here are also formally defined as sequences of integers; however, the combinatorial demands which combinations of integers give a valid position sequence have, or course, be adapted to the different combinatorial structure that our word sequence in this paper do have. On the other hand, both types of position sequences have in common that they give precisely the valid split-ups of word sequences in such subword sequences that again satisfy all combinatorial demands of a word sequence. However, since here the internal combinatorial structure of a word sequence is not an insertion structure, but a substitution structure, and since the substitutes of two letters can mix (even though the given two letters have clearly been placed one after the other, cf. 0.7(4.)), a full understanding of the phenomena of the combinatorics of position sequences takes some effort. The combinatorial rules are not to be exposed in this paper; it should be clear that they are finite and well-defined since word sequences have these properties. However, we wish to point out that the entries of two position sequences can on a certain level satisfy the relation "$<$", and on another level satisfy the relation "$>$". This needs to be understood, when it comes to defining the order of position sequences. Further aspects of positions sequences that need to be discussed in a full exposition (cf. [Zas0].§C.5)) are the facilities for extending the place before or after the letter of a word sequence to a position sequence, the relation between position sequences and the parameter values of a path, and, in particular, how those position sequences that correspond to the start- and endpoints of an act of crossing are related to the corresponding letter in a word sequence. Since the realization mapping is compatible between the possible split-ups of paths and of word sequences, position sequences can be associated to parameter values, and hence there exists a natural order, at least on the appropriate equivalence classes of position sequences. However, the combinatorial aspects of this order and the equivalence relation are amongst a huge

number of little details which need to be understood and hence are discussed in [Zas0] — and which make that paper so long.

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