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Arithmetic of Complex Sets*

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Abstract — Zusammenfassung

Arithmetic of Complex Sets. Let $\mathbb{I}(\mathbb{R})$ be the set of all real closed intervals and let $\Omega_1 := \{+, -, \times, /\}$ be the set of arithmetic operators on $\mathbb{R}$. By extending Ω_1 from $\mathbb{R}$ to $\mathbb{I}(\mathbb{R})$ as usual one finds that $\mathbb{I}(\mathbb{R})$ is closed with respect to the operations from Ω_1 (R. E. Moore [9]). In the literature several possibilities are discussed to go over from complex numbers to "complex intervals": rectangles (Alefeld [1] et al.), discs (Henrici [4] et al.) or ellipses (Kahan [5] et al.). In all three cases the resulting sets are not closed with respect to Ω_1, since the multiplication and division of such "intervals" does not lead to sets of the same kind. In what follows the question is treated whether there are classes of complex sets ("generalized intervals") which are closed with respect to Ω_1 or to subsets of Ω_1. One such class is easy to find. Also the shape of the sets involved is discussed. If it is assumed however that the sets under consideration are described by a finite number of parameters then there is *no* such class closed under Ω_1.

Zur komplexen Mengen-Arithmetik. Es sei $\mathbb{I}(\mathbb{R})$ die Menge reeller abgeschlossener Intervalle und $\Omega_1 := \{+, -, \times, /\}$ die Menge der arithmetischen Operationen auf $\mathbb{R}$. Erweitert man dann Ω_1 von $\mathbb{R}$ auf $\mathbb{I}(\mathbb{R})$ wie üblich, dann ist $\mathbb{I}(\mathbb{R})$ abgeschlossen gegenüber den Operationen von Ω_1 (R. E. Moore [9]). In der Literatur werden verschiedene Möglichkeiten vorgeschlagen, um von komplexen Zahlen zu „komplexen Intervallen“ überzugehen: Rechtecke (Alefeld [1] et al.), Kreise (Henrici [4] et al.), Ellipsen (Kahan [5] et al.). In allen drei Fällen sind die entstehenden Mengen nicht mehr abgeschlossen gegenüber Ω_1, weil die Multiplikation und Division solcher „Intervalle“ nicht wieder auf Mengen derselben Art führt. Im folgenden wird die Frage behandelt, ob es Klassen von komplexen Mengen („verallgemeinerte Intervalle“) gibt, die abgeschlossen sind gegenüber Ω_1 oder Teilmengen von Ω_1. Außerdem wird untersucht, welche „Gestalt“ solche Mengen besitzen. Während man solche Klassen sofort angeben kann, wird sich zeigen lassen, daß die Abgeschlossenheit *nicht* mehr erreichbar ist, wenn man noch zusätzlich fordert, daß diese Mengen (nur) durch endlich viele Parameter beschrieben werden.

1. Introduction and Formulation of the Problem

Let $S := \{a, b, \ldots\}$ be a base set with the power set $\mathbb{P}(S) := \{A, B, \ldots\}$. Let there exist unary and binary operators ω_ν defined on S and $S \times S$; let $\Omega := \{\omega_1, \omega_2, \ldots\}$ be the set of these operators. Assume that S is *closed* (invariant) under each

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operation $\omega \in \Omega$, i.e. let[1]

$$a \,\omega\, b \in S \quad \text{for all} \quad a, b \in S,\ \omega \in \Omega.$$

One defines as usual

Definition (set operator): Let $A, B \in \mathbb{P}(S)$. Define for all $\omega \in \Omega$

$$A \,\omega\, B := \{a \,\omega\, b \mid a \in A,\ b \in B\}^{1}. \tag{1}$$

Properties:

1. The operators ω defined by (1) on $\mathbb{P}(S) \times \mathbb{P}(S)$ are *extensions* of the operators ω on $S \times S$. Hence the same symbol ω can be used.
2. The operators ω defined by (1) are *inclusion isotone* on $\mathbb{P}(S) \times \mathbb{P}(S)$, i.e. for all $A, B, C, D \in \mathbb{P}(S)$ and all $\omega \in \Omega$ the following is true:

$$C \subseteq A,\ D \subseteq B \text{ implies } C \,\omega\, D \subseteq A \,\omega\, B. \tag{2}$$

3. $\mathbb{P}(S)$ is *closed* with respect to all $\omega \in \Omega$.

Hence the following three questions can be asked:

i) Let the set M be given such that $S \subseteq M \subseteq \mathbb{P}(S)$. Under which sets Ω of operators is M closed?

ii) Let the set of operators Ω be given. Is there a non trivial set M such that $S \subseteq M \subseteq \mathbb{P}(S)$ which is closed with respect to Ω?

iii) Let ii) be satisfied. What is the shape of the elements of M?

In what follows some already known results for answering i) are summarized. Subsequently the questions ii) and iii) will be partially answered for $S := \mathbb{C}$ and

$$\Omega_1 := \{+, -, \times, /\}. \tag{3}$$

To this end, the following sets will be used as examples:

Special case: *Intervals and balls on S.*

Let $\leqq$ be an order relation and/or let $|\,.\,,.\,|$ be a distance function defined on S.

Definition (intervals, balls): Let $\mathbb{I}(S)$ denote the set of all *intervals*

$$A = [\underline{a}, \bar{a}] := \{x \in S \mid \underline{a} \leqq x \leqq \bar{a}\},$$

where $\underline{a}, \bar{a} \in S$, $\underline{a} \leqq \bar{a}$. Let $\mathbb{K}(S)$ denote the set of all *balls*

$$A := \{x \in S \mid |a, x| \leqq \alpha\},$$

where $a \in S$, $0 \leqq \alpha \in \mathbb{R}$.

Clearly $S \subseteq \mathbb{I}(S)$, $\mathbb{K}(S) \subseteq \mathbb{P}(S)$.

Example: Let $S := \mathbb{C}$. By using the componentwise partial ordering together with the Euclidean distance one sees: $\mathbb{I}(\mathbb{C})$ are exactly the rectangles with sides parallel to the coordinate axes while $\mathbb{K}(\mathbb{C})$ are all the discs.

[1] For simplicity this is written down only for binary operators.

2. Known Results

Let the arithmetic operator symbols $+$, $-$, $\times$, $/$ have the usual meaning. By $\times\mathbb{R}$ and $\times\mathbb{C}$ is meant that the elements under consideration are multiplied only by real and complex numbers. Let the symbol $1/$ denote the unary operator: inversion. Here and through the whole paper it is explicitly assumed that the division $/$ is never by zero nor by a set containing zero. Sets containing the infinite point are therefore excluded. Such "extended" intervals have been introduced for $S := \mathbb{R}$ for the first time by W. M. Kahan [6] and have been implemented by S. E. Laveuve [8].

The following Table 1 answers the question i) for some sets M:

Table 1. *Base sets S, sets M and operator sets Ω for which M is closed*

S	M	Ω	Literature
$\mathbb{R}$	$\mathbb{I}(\mathbb{R})$	$\Omega_1 := \{+, -, \times, /\}$	Moore [9]
	$\mathbb{K}(\mathbb{R})$		
$\mathbb{C}$	$\mathbb{I}(\mathbb{C})$	$\{+, -, \times\mathbb{R}, \mathrm{Re}, \mathrm{Im}\}$	Alefeld [1]
	$\mathbb{K}(\mathbb{C})$	$\{+, -, \times\mathbb{C}, 1/\}$	Henrici [4], Krier [7], Hauenschild [3]
$\mathbb{R}^n$	$\mathbb{I}(\mathbb{R}^n)$	$\{+, -, \times\mathbb{R}\}$	Alefeld-Herzberger [2]
	$\mathbb{K}(\mathbb{R}^n)$		
$C[a, b]$	$\mathbb{I}(C[a, b])$	$\Omega_1 := \{+, -, \times, /\}$	
	$\mathbb{K}(C[a, b])$		

In $\mathbb{R}$ order relation and distance are defined as usual. In $\mathbb{C}$ and $\mathbb{R}^n$ the componentwise partial ordering and the Euclidean distance are used. In the space $C[a, b]$ of continuous functions on the interval $[a, b] \in \mathbb{I}(\mathbb{R})$ the pointwise order relation and distance are taken from $\mathbb{R}$.

It is well known that neither $\mathbb{I}(\mathbb{C})$ nor $\mathbb{K}(\mathbb{C})$ are closed with respect to Ω_1 from (3). In order to overcome that difficulty a special *rectangle arithmetic* and several *disc arithmetics* have been introduced, see [1], [4], [7], [3]. In extension of the disc arithmetic an *ellipse arithmetic* has been introduced by W. M. Kahan [5]. This has been done because the set of all ellipses in $\mathbb{C}$ (which are not treated here) are also not closed with respect to Ω_1 under (3). — These problems are not at all trivial. This is shown by the definition of an "optimal" disc arithmetic by N. Krier [7] which turned out *not* to be inclusion isotone as defined in (2), see [7], [3].

3. The Set M_1

For the rest of the paper $S := \mathbb{C}$ is considered. For $\Omega_1 := \{+, -, \times, /\}$ [2] the above question ii) can immediately be answered by use of the recursive

Definition (set M_1):

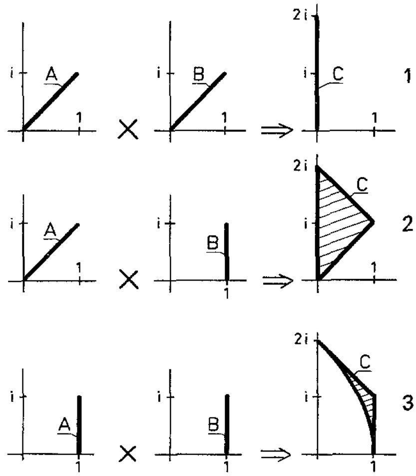
a) Let $\mathbb{C} \subseteq M_1$.

b) Let $I := [-1, +1] \in M_1$.

c) If $A, B \in M_1$ then let also be $A \,\omega\, B \in M_1$ for all $\omega \in \Omega_1$ [2].

By definition M_1 is closed under Ω_1, furthermore $\mathbb{C} \subseteq M_1 \subseteq \mathbb{P}(\mathbb{C})$ where $\mathbb{C} \neq M_1$ and $M_1 \neq \mathbb{P}(\mathbb{C})$. Hence M_1 is a solution to the problem in question ii).

To show the variety of the geometric shapes of the elements in M_1 in response to question iii), some figures will be shown. The interval $I := [-1, 1]$ can be turned, expanded and translated into any arbitrary position just by multiplication by a complex constant and/or addition to a complex number. The multiplication of two segments may be a segment again (see Fig. 1), a twodimensional set containing inner points which is convex (see Fig. 2, triangle) or not convex (see Fig. 3).



Figs. 1 to 3. Multiplication of two segments A and B. The resulting set $C := A \times B$ is a segment, a triangle or a non-convex set

Obviously $\mathbb{I}(\mathbb{R}) \subseteq M_1$ is true by definition of M_1. Furthermore $\mathbb{I}(\mathbb{C}) \subseteq M_1$ as is indicated by Fig. 4.

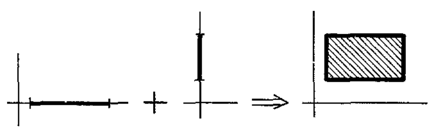


Fig. 4. Adding two segments on the coordinate axes produces an arbitrary rectangle with sides parallel to the axes

[2] The division by zero or by a set containing zero is always excluded.

It is not so obvious however, that $\mathbb{K}(\mathbb{C}) \subseteq M_1$ is also true. This can be seen by the sketch in Fig. 5.

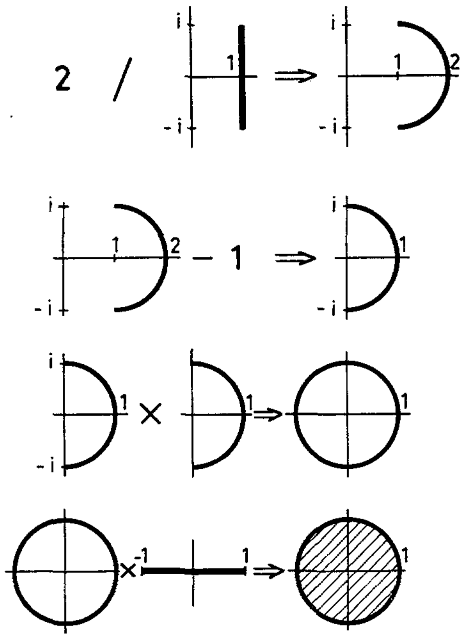


Fig. 5. How to produce the unit circle and the unit disc from numbers and segments using arithmetic operations

By algebraic formulae this means: The boundary ∂K of the unit disc $K := \{z \in \mathbb{C} \mid |z| \leqq 1\}$ can be written as

$$\partial K = (2/(1 + i \times I) - 1) \times (2/(1 + i \times I) - 1),$$

where $I := [-1, 1]$. Hence $K = \partial K \times I$.

Therefore the two theorems are true:

Theorem 1: *Assume that* $\mathbb{C} \subseteq M$, $\mathbb{I}(\mathbb{R}) \subseteq M \subseteq \mathbb{P}(\mathbb{C})$ *and suppose that the set* M *is closed under*

$$\Omega_2 := \{+, -, \times \mathbb{C}\}.$$

Then $\mathbb{I}(\mathbb{C}) \subseteq M$, *but* $M \neq \mathbb{I}(\mathbb{C})$.

Theorem 2: *Assume that* $\mathbb{C} \subseteq M$, $\mathbb{I}(\mathbb{R}) \subseteq M \subseteq \mathbb{P}(\mathbb{C})$ *and suppose that the set* M *is closed with respect to* Ω_1 *defined by* (3). *Then* $\mathbb{K}(\mathbb{C}) \subseteq M$, *but* $M \neq \mathbb{K}(\mathbb{C})$.

In Figs. 6 and 7 there are two more examples showing the self multiplication of a special rectangle and of a disc. The resulting set in Fig. 7 has a cardioid as boundary, see [7].

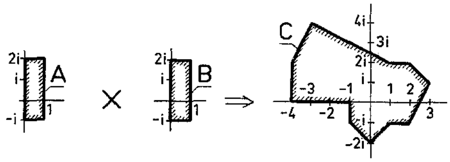


Fig. 6. Multiplication of two rectangles A and B with $A=B$. The result $C:=A\times B$ has 11 corners. Three of them are reentering, one with the angle $\pi/2$, one with $3\,\pi/4$ and the last with $\pi-\arctan\,.5$

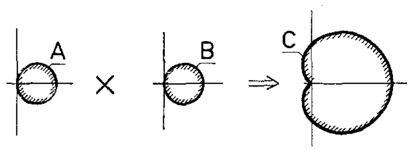


Fig. 7. Multiplication of two discs A and B with $A=B$. The resulting set $C:=A\times B$ is bounded by a cardioid. See N. Krier [7], page 26

4. Non-Existence of a Set M Defined by Finitely Many Parameters Which Is Closed Under Ω_1 or Ω_2

Any real interval $A=[\underline{a},\bar{a}]\in\mathbb{I}(\mathbb{R})$ is completely determined by the two real parameters $\underline{a}$, $\bar{a}$. Four real parameters are sufficient for the description of the complex intervals in $\mathbb{I}(\mathbb{C})$ while only three real parameters suffice for the discs in $\mathbb{K}(\mathbb{C})$. Any set to be used for practical purposes has to have the property that it can be described by a finite number of parameters. Unfortunately, the set M_1 does *not* have that feature. In order to show that, first a

Definition (admissible set M): The set $M\subseteq\mathbb{P}(\mathbb{C})$ is called *admissible* if there is an integer m such that for each element $A\in M$ the following is true: there are at most m complex numbers $a_\nu\in\mathbb{C}$ for $\nu=1\,(1)\,m$ which are called "corners" of A such that $a_\nu\in\partial A$. Let ∂A between two corners consist of a smooth (2 times continuous differentiable) Jordan curve which is called a "side". Assume furthermore that the boundary ∂A of A has no double points.

It is important that the natural number m is the same for all elements $A\in M$. In this definition nothing is said about the describability of the sides by a finite number of parameters. — The sets $\mathbb{I}(\mathbb{C})$ and $\mathbb{K}(\mathbb{C})$ are obviously admissible with $m=2$ for $\mathbb{I}(\mathbb{C})$ and $m=0$ for $\mathbb{K}(\mathbb{C})$. With this definition one gets immediately

Theorem 3: *There is no admissible set $M\subseteq\mathbb{P}(\mathbb{C})$ with the properties $\mathbb{C}\subseteq M$ and $\mathbb{I}(\mathbb{R})\subseteq M$ which is closed under*

$$\Omega_2:=\{+,\,-,\,\times\mathbb{C}\}.$$

Conclusions:

1. The set M_1 is not admissible.
2. Consequently there is even no such set which is closed under

$$\Omega_1:=\{+,\,-,\,\times,/\}.$$

Proof: Define the sets

$A_1 := I := [-1, 1]$,
$A_2 := i \times I$,
$A_\nu := e^{i\varphi_\nu} \times I \quad$ for $\nu = 2, 3, \ldots$.

Here all the angles $\varphi_1 := 0$, $\varphi_2 := \pi/2$, $0 < \varphi_\nu < \pi$ for $\nu = 2, 3, \ldots$ are chosen such that $\varphi_\nu \neq \varphi_\mu$ for $\nu \neq \mu$ and $\nu, \mu = 1, 2, \ldots$. Define, furthermore,

$B_1 \quad := A_1$,
$B_{\nu+1} := B_\nu + A_{\nu+1} \quad$ for $\nu = 1, 2, \ldots$.

From the definition (1) in the case of addition one derives immediately the following facts:

B_1 is a segment,
B_2 is a square,
B_3 is a hexagon,
B_4 is an octogon,

see Fig. 8. In general: B_ν is a polygon having 2ν corners.

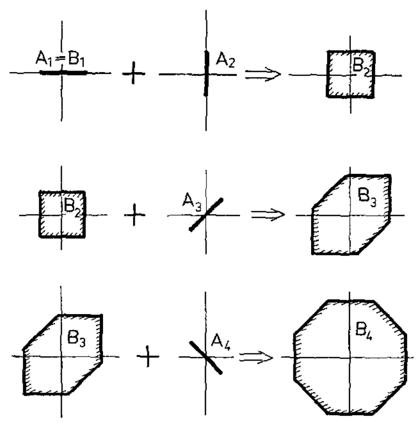


Fig. 8. The construction of the sets B_1 to B_4

The sets B_ν so produced are admissible for $2\nu \leq m$, but this is not the case for the whole sequence $\{B_\nu\}$.

5. Concluding Remarks

In Theorem 3 the only "real" assumption made was $\mathbb{I}(\mathbb{R}) \subseteq M$. It is an interesting question to ask if the negative result of Theorem 3 remains true if this assumption is abandoned. Moreover, one can even think of prohibiting sets $A \in M$ which have corners. A prototype of such a set would be $\mathbb{K}(\mathbb{C})$. Such sets M are probably not very useful for practical purposes. One can however suspect that this abandonment would not bring much. More precisely, one states the following

Conjecture: The result of Theorem 3 for Ω_1 instead of Ω_2 remains valid for $M \neq \mathbb{C}$ even if $\mathbb{I}(\mathbb{R}) \nsubseteq M$ is permitted.

Plausibility consideration: Let $A \in M$ and let M be an admissible set. The sides of A are smooth Jordan curves, moreover ∂A has no double points. It is no loss of generality to assume that A has interior points as can be seen by the examples of section 3. Suppose $a \in \partial A$ is not a corner and suppose that at the point a, the side of A separates the interior of A from $\mathbb{C} \backslash A$. Then it is possible to approximate a from the interior and from the exterior by two nondegenerate discs K_1 and K_2 such that $K_1 \subseteq A$ and that $A \cap K_2 = a$. By using translations, turns, expansions and eventually an inversion it is always possible to achieve the situation which is sketched in Fig. 9: Here $a=0$, the two circles ∂K_1 and ∂K_2 have vertical tangents at a, furthermore $K_1 \subseteq A \subseteq K_2$ and Re $K_2 \geqq 0$.

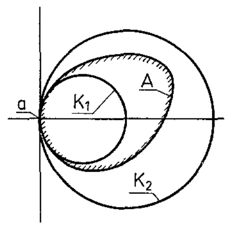


Fig. 9

By performing the self multiplications $B := K_1 \times K_1$, $C := A \times A$, $D := K_2 \times K_2$ one gets the situation sketched in Fig. 10: The two sets B and D are cardioids and both have a cusp at the origin.

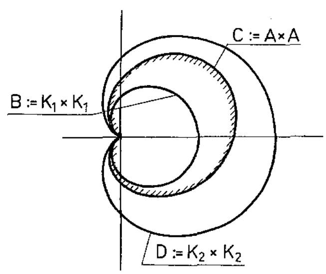


Fig. 10

Because of the inclusion isotonicity (2) of multiplication, the inclusion $B \subseteq C \subseteq D$ holds, hence C has a cusp there too.

If M is closed with respect to Ω_1 then $C \in M$, therefore M contains elements with (inner) cusps. If one adds or subtracts such sets consecutively one gets more sets $\in M$ which have more and more (inner) corners, see Fig. 11 (to show the behaviour more clearly the cardioid was replaced in Fig. 11 by a set C which consists piecewise of circles). The inner angle of the cusp of C is 2π. By adding/subtracting one gets angles $< 2\pi$. But it can easily be seen for special sets C that these angles remain always $> \pi$. It is the guess of the author that this behaviour is generally true. In that case, starting with C, one could produce new elements in M with arbitrarily many corners. Hence M could not be admissible which is the conjecture.

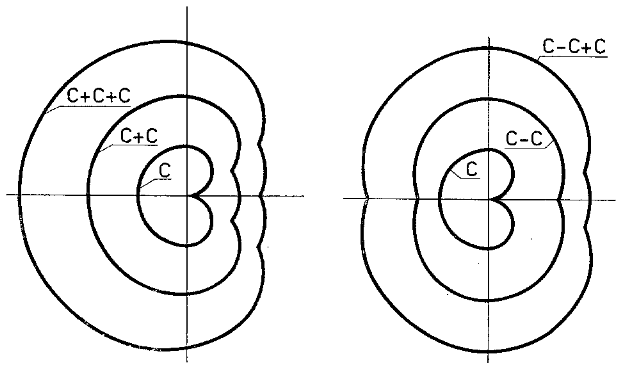


Fig. 11. The sets $C+C$, $C+C+C$, $C-C$ and $C-C+C$ to a set C bounded by circular arcs with a cusp

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