THE SZEGÖ KERNEL – BASIC DEFINITIONS AND OBJECTS (LECTURE NOTES)

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We are interested in the asymptotic of sequences of representations $\rho_N : G \to \mathrm{GL}(W_N)$ of a Lie group G with $\dim W_N \to \infty$ for $N \to \infty$. A natural choice for such a sequence is the following: Let M be a manifold with symmetries, i.e., M is equipped with an action of a Lie group G, and let L be a line bundle on M. For any $N \in \mathbb{N}$ the G action on the manifold M induces a linear representation of G on the vector spaces $W_N = \Gamma(M, L^N)$, which is the vector space of sections from M into the N-th power L^N of the line bundle $L \to M$.

We consider the simplest case, where M is a complex projective manifold, $M \subset \mathbb{P}(V)$, and the holomorphic action action of G on M comes from a regular representation $G \to \mathrm{GL}(V)$ which induces a G-action on $\mathbb{P}(V)$ and on any G-invariant subset $M \subset \mathbb{P}(V)$. Furthermore, we will assume that G is abelian (and connected), i.e., $G = (\mathbb{C}^*)^m$, $m = \dim M$, and the action of the maximal compact subgroup $(S^1)^m \subset (\mathbb{C}^*)^m$ is compatible with a Kähler form ω on M where ω is the pull back of the Fubini-Study form on $\mathbb{P}(V)$ to the complex submanifold M.

Remark 1. We consider submanifolds M of a projective space because for a Hilbert space V, the projective space $\mathbb{P}(V)$ is the state space (vectors of length 1 modulo phase). The restriction to the submanifold M corresponds to further conditions imposed on the states. Our particular choice of M, G and G simplifies the situation as follows:

- M is the zero set of homogeneous polynomials is $\mathbb{P}(V)$, G-action is algebraic
- There are finiteness results for holomorphic line bundles on projective varieties. In particular, the vector spaces of section are finite dimensional, $\dim \Gamma(M, L^N) < \infty$.
- If $G = (\mathbb{C}^*)^m$ then any G-representation decomposes into a direct sum of 1-dimensional eigenspaces. This means that the representation can be conveniently described by the weights and the multiplicities.
- We assume $m = \dim M$. More precisely, M is the closure of a G-orbit, $M = \overline{G.[v]}$ for some $v \in V$ so that G acts freely on a dense, Zariski-open subset $G.[v] \subset M$.
- The existence of a Kähler form ω compatible with the action of the compact subgroup $(S^1)^m \subset (\mathbb{C}^*)^m$ allows us to identify the space of sections $\Gamma(M, L^N)$ with a Hilbert space of functions on a real manifold X associated to M. More precisely, $V \setminus \{0\} \supset X \to \mathbb{P}(V)$ is an S^1-bundle, i.e., the manifold X consists of unit vectors in phase space. The representation spaces W_N can be considered as subspaces of functions on X satisfying an equivariance condition.

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1. Line bundles on projective manifolds

Let V be a complex vector space. The *projective space* $\mathbb{P}(V)$ is the set of 1-dimensional subspaces and the basic example of a complex projective manifold. Since every $v \in V \setminus \{0\}$ lies in exactly one such subspace, there is a canonical projection $\pi_0 : V \setminus \{0\} \to \mathbb{P}(V)$, $v \mapsto [v]$. We can identify the projection $\pi_0 : V \setminus \{0\} \to \mathbb{P}(V)$ with the quotient of $V \setminus \{0\}$ by the $\mathbb{C}^*$-action $\mathbb{C}^* \times V \to V$, $(\lambda, v) \mapsto \lambda v$, since $\pi_0(v) = \pi_0(v')$ iff $v' = \lambda v$ for some $\lambda \in \mathbb{C}^*$. The following covering by local coordinates gives $\mathbb{P}(V)$ the structure of a complex manifold such that π_0 is a holomorphic map.

1.1. **The complex manifold $\mathbb{P}(V)$.** Let $\{e_1, \dots, e_n\}$ be a basis of the complex vector space V and let $(v_1, \dots, v_n)$ be coordinates on the complex vector space such that $v = \sum_{j=1}^n v_j(v) e_j$ for all $v \in V$. The sets $U_j := \{[v] : v_j(v) \neq 0\}$ are well-defined open subsets of $\mathbb{P}(V)$ for $j = 1, \dots, n$ and $\cup_{j=1}^n U_j = \mathbb{P}(V)$. The standard choice of local coordinates on U_j is given by:

$$\varphi_j : U_j \to \mathbb{C}^{n-1}, \quad [v_1 : \dots : v_n] \mapsto \left(\frac{v_1}{v_j}, \dots, \frac{v_{j-1}}{v_j}, \frac{v_{j+1}}{v_j}, \dots, \frac{v_n}{v_j} \right)$$

One checks that the coordinate changes $\varphi_k \circ \varphi_j^{-1} : \varphi_j(U_j \cap U_k) \to \varphi_k(U_j \cap U_k)$ and π_0 are holomorphic maps. Note that $U_k \cap U_j \cong \mathbb{C}^* \times \mathbb{C}^{n-2}$ for $j \neq k$.

1.2. **Line bundles over $\mathbb{P}(V)$.** Every fibre of the projection $\pi_0 : V \setminus \{0\} \to \mathbb{P}(V)$ is isomorphic to $\mathbb{C}^*$. Furthermore, the open sets $\pi_0^{-1}(U_j) = \{v : v_j(v) \neq 0\} \subset V \setminus \{0\}$ are isomorphic to $U_j \times \mathbb{C}^*$ where the isomorphism is $\phi_j : U_j \times \mathbb{C}^* \to \pi_0^{-1}(U_j)$ can be described most easily by $\phi_j^{-1} : v \mapsto (\pi_0(v), v_j(v))$. Note that $\pi_0 \circ \phi_j([v], \lambda) = [v]$ for all $[v] \in U_j$ and $\lambda \in \mathbb{C}^*$. This means that $\pi_0 : V \setminus \{0\} \to \mathbb{P}(V)$ is a fibre bundle with fibre $\mathbb{C}^*$.

Remark 2 (Definition of a fibre bundle). Recall that a map $\pi : E \to B$ is called a fibre bundle with fibre F if the base space B can be covered by open sets U_α such that there exist isomorphisms $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ satisfying $\pi \circ \phi_\alpha(b, f) = b$ for all $b \in U_\alpha$ and $f \in F$.

The basic example of a fibre bundle $\pi_0 : V \setminus \{0\} \to \mathbb{P}(V)$ contains even more structure. Consider the open sets $U_{jk} := U_j \cap U_k$ and the restrictions to $U_{jk} \times \mathbb{C}^*$ of the local trivializations ϕ_j and ϕ_k. Note that

$$\begin{aligned} \phi_k^{-1} \circ \phi_j([v], z) &= \phi_k^{-1} \left(z\frac{v_1}{v_j}, \dots, z\frac{v_{-1}}{v_j}, z, z\frac{v_{j+1}}{v_j}, \dots, z\frac{v_n}{v_j} \right) \\ &= \left([v], \frac{v_k}{v_j} z \right) = ([v], g_{kj}([v])z) \end{aligned}$$

for all for all $[v] \in U_{jk}$ where $[v] = [v_1 : \dots : v_n]$ and $g_{kj} : U_{jk} \to \mathbb{C}^*$, $[v] \mapsto v_k/v_j$. The fibre bundle is completely described by the transition functions g_{kj} that satisfy compatibility conditions $g_{lk} g_{kj} = g_{lj}$ on $U_{jkl} := U_j \cap U_k \cap U_l$. The group $G = \mathbb{C}^*$ acts by multiplication on the fibre $F = \mathbb{C}^*$ and $\pi_0 : V \setminus \{0\} \to \mathbb{P}(V)$ is an example of a G-bundle with fibre F (and $G = F$).

Remark 3 (Definition of a G-bundle). Recall that a fibre bundle $\pi : E \to B$ with fibre F is called a G-bundle if G acts on F and there exist transitions functions $g_{\beta\alpha} : U_\alpha \cap U_\beta \to G$ such that the change of local trivialization can be described by the transition functions, i.e., $\phi_\beta^{-1} \circ \phi_\alpha(b, f) = (b, g_{\beta\alpha}(b).f)$ for all $b \in U_\alpha \cap U_\beta$ and $f \in F$. The compatibility conditions, called cocycle conditions, follow directly from the properties of a group action. A covering $\{U_\alpha\}$ of the base B, a fibre F on which a group G acts and a set of transition functions $g_{\beta\alpha} : U\alpha \cap U_\beta \to G$ that satisfy the cocycle condition determine uniquely a G-bundle with fibre F over B.

If in every fibre $F = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ of the $\mathbb{C}^*$-bundle $\pi_0 : V \setminus \{0\} \to \mathbb{P}(V)$ zero is added, this gives rise to a new bundle $\mathcal{O}(-1) \to \mathbb{P}(V)$ which is called the *tautological bundle*. The tautological bundle is a $\mathbb{C}^*$-bundle with fibre $F = \mathbb{C}$ over $\mathbb{P}(V)$ that has the same transition functions as $\pi_0 : V \setminus \{0\} \to \mathbb{P}(V)$. It is the first example of a *line bundle* on $\mathbb{P}(V)$.

Remark 4 (Definition of vector bundle and line bundle). A G-bundle $\pi : E \to B$ with fibre F is called a vector bundle if F is a vector space and $G = \mathrm{GL}(F)$. The basic examples of vector bundles are the tangent bundle and the cotangent bundle. If $\dim F = 1$ then the bundle is called a line bundle. For a complex line bundle the structure group G is $\mathbb{C}^*$.

For every $N \in \mathbb{Z}$ the line bundle $\mathcal{O}(-N) \to \mathbb{P}(V)$ is defined by the transition functions $g_{kj} : U_{jk} \to \mathbb{C}^*$, $[v] \mapsto (v_k/v_j)^N$. The bundle $\mathcal{O}(1) \to \mathbb{P}(V)$ is the bundle dual to $\mathcal{O}(-1)$. It is called the *hyperplane section bundle* and is denoted by H.

Remark 5 (n-th power of a line bundle). Given any line bundle $\pi : L \to M$ with transition functions $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ we define, for every $N \in \mathbb{Z}$, the N-th power L^N to be the line bundle over M with transition functions $g_{\beta\alpha}^N$. Since $\mathbb{C}^*$ is commutative, the transition functions $g_{\beta\alpha}^N$ satisfy the cocycle condition too.

Theorem 1. *Any holomorphic line bundle on $\mathbb{P}(V)$ is isomorphic to $\mathcal{O}(N)$ for some $N \in \mathbb{Z}$.*

1.3. **Duality between the tautological bundle H^{-1} and the hyperplane section bundle H.** The tensor product of the hyperplane section bundle $H \to \mathbb{P}(V)$ and the tautological bundle $H^{-1} \to \mathbb{P}(V)$ is isomorphic to the trivial bundle, because its transition functions are $g_{\alpha\beta}^{-1} g_{\alpha\beta} \equiv \mathrm{Id}$. Using this isomorphism a point x in the fibre of the line bundle $H^{-1} \to \mathbb{P}(V)$ over $[v]$ can be regarded as a complex linear function f_x on the fibre of the line bundle $H \to \mathbb{P}(V)$ over $[v]$.

1.4. **Sections.** In general, a *section* of a fibre bundle $\pi : E \to B$ is a map $s : B \to E$ such that $\pi \circ s = \mathrm{Id}_B$. We denote the set of holomorphic sections of a holomorphic line bundle $\pi : L \to M$ by $\Gamma(L, M)$. For a vector bundle, the set of sections forms a vector space. In particular, $\Gamma(L, M)$ is a complex vector space. Note that for any vector bundle there exits at least one section, the zero section, defined locally by $s_\alpha : U_\alpha \to U_\alpha \times F$, $b \mapsto (b, 0)$, where $0 \in F$ is the origin of the vector space F.

Remark 6. Locally sections of a line bundle are just functions from U_α to $\mathbb{C}$. But on a non-trivial bundle, due to the transition functions, they are not restrictions of globally defined functions. For example, the only holomorphic functions on $\mathbb{P}(V)$ are the constants, but the line bundles $\mathcal{O}(N)$ have many sections for $N > 0$.

Remark 7. On G-principal bundles, i.e., $F = G$, the existence of a section implies the triviality of the bundle.

Let us calculate $\Gamma(\mathcal{O}(N), \mathbb{P}(V))$. Let s be a section of $\mathcal{O}(N) \to \mathbb{P}(V)$. Then $s_j := \phi_j \circ s \circ \varphi_j^{-1} : U_j \to U_j \times \mathbb{C}$ is a section of the trivial bundle $U_j \times \mathbb{C} \to U_j$, hence, an element of $\mathbb{C}[v_1/v_j, \ldots, v_n/v_j]$. Now, $s_j g_{jk} = s_k$ with the transition functions $g_{jk} = (v_k/v_j)^{-N}$ and

$$s_j(v_l/v_j)\left(\frac{v_k}{v_j}\right)^{-N} = s_j(v_l/v_j)\left(\frac{v_j}{v_k}\right)^{N} = s_k(v_l/v_k)$$

for polynomials s_j and s_k implies $\deg s_j, \deg s_k \leq N$.

Theorem 2. *The sections of the N-th power of the hyperplane section bundle can be identified with the homogeneous polynomials of degree N in $n+1$ variables:*

$$\Gamma(\mathbb{P}(V), H^N) = \begin{cases} \{0\} & n < 0 \\ \mathbb{C}[V]_{(N)} & n \geq 0 \end{cases},$$

where $\mathbb{C}[V]_{(N)}$ denotes the subspace of homogeneous polynomials of degree N.

1.5. **Complex projective manifolds.** If the common zero set

$$M := \{f_1 = \ldots = f_r = 0\} \subset \mathbb{P}(V)$$

of a finite set of homogeneous polynomials $f_j \in \mathbb{C}[V]_{(d_j)}$ of degree d_j is smooth, then M is called *complex projective manifold*. The theorem of Chow states that any holomorphic submanifold of $\mathbb{P}(V)$ is in fact a smooth subvariety, i.e., given as the zero set of a finite set of homogeneous polynomials.

A holomorphic line bundle L on a complex manifold M is given by a covering $\{U_\alpha\}$ and a set of transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ satisfying the cocycle (or compatibility conditions) $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$ for all indices α, β, γ.

Remark 8. Since isomorphism classes of holomorphic line bundles on a complex manifold M are classified by $H^1(M, \mathcal{O}^*)$ any holomorhic line bundle on a complex projective manifold comes from an algebraic line bundle on the corresponding complex algebraic variety.

We consider complex projective manifolds $M \subset \mathbb{P}(V)$ and line bundles $L \to M$ with transitions functions g_α, that are just restrictions to M of the transition functions of the hyperplane bundle $H \to \mathbb{P}(V)$. Then $L^N \to M$ is just the pull back of $H^N \to \mathbb{P}(V)$ by the inclusion $M \to \mathbb{P}(V)$.

Remark 9. A line bundle $L \to M$ is called *ample* if for some $k \in \mathbb{N}$ the map $\Phi : M \to \mathbb{P}(\Gamma(M, L^k))$, $p \mapsto [s_0(p) : \ldots : s_K(p)]$, embeds M into $\mathbb{P}(V)$ such that L^k is pull-back of the hyperplane section bundle on $\mathbb{P}(\Gamma(M, L^k))$.

1.6. **Properties of projective line bundles.** Since a complex projective manifold is a compact manifold and a line bundle is a coherent sheaf, the vector space of sections $\dim \Gamma(M, L^N)$ is *finite dimensional* for all $N \geq 0$.

Recall that $\Gamma(\mathbb{P}(V), H^N) \cong \mathbb{C}[V]_{(N)}$. Hence, with $n = \dim V$

$$\dim \Gamma(\mathbb{P}(V), H^N) = \binom{N+n}{n} = \prod_{j=1}^{N} \frac{j+n}{j} = \prod_{j=1}^{N} \left(1 + \frac{n}{j}\right)$$
$$< \dim V \sum_{j=1}^{N} \frac{1}{j}$$

and $\lim_{N\to\infty} \dim \Gamma(\mathbb{P}(V), H^N) = \infty$, since the harmonic series $\sum_{j=1}^{\infty} 1/j$ diverges.

If $L \to M$ is an ample line bundle, then $\dim \Gamma(M, L^N) \to \infty$ as $N \to \infty$.

2. The complex-real manifold X

Given a basis of the complex vector space V there is the canonical hermitian inner product $\langle\,,\,\rangle : V \times V \to \mathbb{C}$ and a norm $||\,|| : V \to \mathbb{R}^{\geq 0}$ with $||v||^2 = \langle v, v\rangle$. We consider the $\mathbb{C}^*$-bundle $\pi_0 : V \setminus \{0\} \to \mathbb{P}(V)$, $v \mapsto [v]$, and the unit sphere $S^{2n-1} = \{v : ||v|| = 1\} \subset V$ which is a $2n-1$-dimensional real manifold. For a complex submanifold $M \subset \mathbb{P}(V)$ let

$$X := \pi_0^{-1}(M) \cap S^{2n-1}.$$

2.1. **The S^1-bundle $X \to M$.** The projection $\pi_0 : X \to N$ is a bundle with fibre S^1. Furthermore, the C^*-action on V by scalar multiplication is transitive on the fibres of $\pi_0 : V \setminus \{0\} \to \mathbb{P}(V)$ and trivial on the base space $\mathbb{P}(V)$. The compact subgroup $S^1 \subset \mathbb{C}^*$ acts transitively on the fibres of $\pi_0 : S^{2n-1} \to \mathbb{P}(V)$. Consequently, there is a natural S^1-action on the S^1-bundle $X \to M$ that is trivial on the base M and transitively in any fiber.

Furthermore, we regard X as a subset of the tautological bundle, because $\pi_0 : V \setminus \{0\} \to \mathbb{P}(V)$ is obtained by removing the zero-section from the tautological bundle $H^{-1} \to \mathbb{P}(V)$.

2.2. **The complex-real structure on X.** For any $x \in X$ the tangent space T_xX is a real vector subspace contained in the complex vector space $T_xV \cong V$ and in the complex vector space $T_x\pi_0^{-1}M \cong T_{\pi_0(x)}M \oplus \mathbb{C}$. There is a maximal complex subspace $T_{x,\mathbb{C}}X \subset T_x\pi_0^{-1}M$ such that $T_xX = T_{x,\mathbb{C}}X \oplus T_x(S^1x)$. This means that X has $\dim_{\mathbb{C}} M$ complex dimensions and one real dimension.

A function f on X is called *holomorphic* if for any $x \in X$ there is a neighbourhood U of x in $\pi_0^{-1}M$ such that f is the restriction of a holomorphic function on U to $X \cap U$.

2.3. **Volume forms on X and M.** Up to a scalar multiple, there exists a unique volume form on the sphere S^{2n-1} which is $\mathrm{SO}_{\mathbb{R}}(V)$ invariant. Since the action of $\S^1 \subset \mathbb{C}^*$ on V is a subgroup of $\mathrm{SO}_{\mathbb{R}}(V)$ and $\mathbb{P}(V) = S^{2n-1}/S^1$, this volume form pushes forward to an invariant volume form on $\mathbb{P}(V)$. By restriction volume forms on $X \subset S^{2n-1}$ and $M \subset \mathbb{P}(V)$ are defined.

2.4. **Holomorphic sections and equivariant functions.** We identify holomorphic section of the line bundles $L^N \to M$ with functions on the manifold X regarding a point $x \in X$ as a functional f_x on fibres of $L \to M$ which is a subset of $H \subset \mathbb{P}(V)$. Consider the map $\Gamma(M, L^N) \to \mathcal{C}^\infty(X)$, $s_N \mapsto \hat{s}_N$ with

$$\hat{s}_N(x) := f_x^{\otimes N}(s_N(\pi_0(x)))$$

where $\pi_0 : X \to M$ and s_N a section of $L^N = L^{\otimes N}$.

We check that the functions $\hat{s}_N$ are equivariant with respect to the S^1-action on X and $\mathbb{C}$ by scalar multiplication:

$$\begin{aligned}\hat{s}_N(e^{i\theta}x) &= f_{e^{i\theta}x}^{\otimes N}(s_N(\pi_0(x))) = \left(e^{i\theta}f_x\right)^{\otimes N}(s_N(\pi_0(x))) = e^{i\theta N}f_x^{\otimes N}(s_N(\pi_0(x))) \\ &= e^{i\theta N}\hat{s}_N(x)\end{aligned}$$

Since the function $\hat{s}_N$ arise from holomorphic sections, they are holomorphic functions on X.

For $N \in \mathbb{N}$ we define $\mathcal{H}^2_N := \{f \in \mathcal{C}^\infty : f \text{ holomorphic}, f(e^{i\theta}x) = e^{i\theta N} f(x)\}$. Let the Hardy space $\mathcal{H}^2$ be the closure of $\oplus_N \mathcal{H}^2_N$ in $L^2(X)$ which is the vector space of square integrable smooth functions on X.

2.5. **Bundle metric.** Using the isomorphism $\Gamma(\mathbb{P}(V), H^N) \cong \mathbb{C}[V]_{(N)}$ the finite dimensional vector spaces $\Gamma(\mathbb{P}(V), H^N)$ can be equipped with a hermitian scalar product. For $s_1, s_2 \in \mathbb{C}[V]_{(N)}$ and $[v] \in \mathbb{P}(V)$ let

$$\langle s_1([v]), s_2([v])\rangle := \frac{s_1(v)\overline{s_2(v)}}{||v||^{2N}} \text{ and } \langle s_1, s_2\rangle := \int_M \langle s_1([v]), s_2([v])\rangle \,\mathrm{dvol}_M .$$

Theorem 3. *The Hilbert spaces $\mathcal{H}^2_N$ and $\Gamma(M, L^N)$ are isomorphic. In particular, the Hardy spaces $\mathcal{H}^2_N$ are of finite dimension.*

3. The Szegö projector

By definition, $\mathcal{H}^2$ is a closed subspace of the Hilbert space $L^2(X)$. We denote the orthogonal projection onto the subspace $\mathcal{H}^2$ by $\Pi : L^2(X) \to \mathcal{H}^2$.

For any unitary basis $\{h_j\}_j$ of the finite dimensional Hilbert space $\mathcal{H}^2_N$ the map

$$f \mapsto \int_X f(y)\Pi_N(\cdot, y)dy$$

defined by the integral kernel

$$\Pi_N(x, y) = \sum_j h_j(x)\overline{h_j(y)}$$

is the orthogonal projection from $L^2(X)$ onto $\mathcal{H}^2_N$. The function $\Pi = \sum_N \Pi_N$ is called *Szegö kernel*.

4. Symmetry – toric varieties

A symmetry of a manifold M refers to an action $G \times M \to M$ of a Lie group G on the manifold M. Here, we assume that this action is defined by a representation of the group G on the vector space V. More precisely, any representation $G \to \mathrm{GL}(V)$ of a group G on a complex vector space V defines an action of G on $\mathbb{P}(V)$ by $g.[v] = [g(v)]$ and a representation of G on the vector space $\mathbb{C}[V]_{(N)}$ for all $N \geq 0$.

If a complex projective submanifold $M \subset \mathbb{P}(V)$ is G-invariant, i.e., $G(M) = M$, then the restrictions to M define an action of G on M and representations of G on $\Gamma(M, L^N)$. Since the G-action on V commutes with the $S^1 \subset \mathbb{C}^*$-action on V by scalar multiplication, which corresponds to $S^1\,\mathrm{Id} \subset \mathrm{GL}(V)$, we obtain representations of the maximal compact subgroup $K \subset G$ on any orthogonal component $\mathcal{H}^2_N$.

If $G = (\mathbb{C}^*)^m$ and $M = \overline{G.[v]}$ for some $[v] \in \mathbb{P}(V)$, then M is called toric variety. In this case, all arising representations of G or its maximal compact subgroup $(S^1)^m$ decompose into direct sums of one-dimensional weight spaces. The idea is to express all associated object like a basis of $\Gamma(M, L^N)$, the bundle metric or the volume form in terms of the weights of the representation of $G = (\mathbb{C}^*)^m$ on V.

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