Explicit Upper Bounds for Dual Norms of Residuals

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Paris / October 13th, 2008

Outline

Background

A posteriori error estimation consists of two steps:

- Prove the equivalence of the primal norm of the error and of the dual norm of the residual.
- Derive easily computable error indicators that yield upper and lower bounds for the dual norm of the residual.

Goal

Derive upper bounds for dual norms of residuals which are:

- fully explicit,
- guaranteed,
- robust.

Tools

- Decomposition of the residual into a regular and singular part. (element and jump residuals)
- Partition of unity based on the nodal shape functions.
- Trace inequalities with explicit constants.
- Poincaré inequalities with explicit constants.

Resumé: Reduce everything to Poincaré inequalities for *convex* domains.

Domain and Norms

- $\Omega \subset \mathbb{R}^d$ open, bounded, connected
- Γ_D Dirichlet boundary
- μ_d, μ_{d-1} Lebesgue measures on $\mathbb{R}^d$ and $\mathbb{R}^{d-1}$
- $L^p(\Omega)$ and $\|\cdot\|_{L^p(\Omega)}$ standard Lebesgue space and norm
- $W^{1,p}(\Omega)$ standard Sobolev space equipped with $\|\,|\nabla\cdot|\,\|_{L^p(\Omega)}$
- $W^{-1,p'}(\Omega)$ dual space equipped with

$$\|\cdot\|_{W^{-1,p'}(\Omega)} = \sup\big\{\langle\cdot, v\rangle : \|\,|\nabla v|\,\|_{L^p(\Omega)} = 1\big\}$$

Partition and Finite Element Spaces

- $\mathcal{T}$ affine equivalent, admissible partition, may be anisotropic
- $\mathcal{N}$, $\mathcal{E}$ vertices and faces corresponding to $\mathcal{T}$
- Σ skeleton (union of all faces) of $\mathcal{T}$
- $S^{1,0}(\mathcal{T})$, $S^{1,0}_D(\mathcal{T})$ standard lowest order conforming finite element spaces
- λ_z, $z \in \mathcal{N}$, standard lowest order nodal shape functions
- ω_z, σ_z union of all elements resp. faces with vertex z
- $\mu_{d,z}$, $\mu_{d-1,z}$, L^p_z, $\|\cdot\|_{L^p_z(\Omega)}$ measures, spaces and norms with weight λ_z, i.e.

$$\|v\|^p_{L^p_z(\Omega)} = \int_\Omega |v|^p \lambda_z$$

Residual

$R \in W^{-1,p'}(\Omega)$ given residual satisfying:

- R has an L^p-representation

$$\langle R, v\rangle = \int_\Omega rv + \int_\Sigma jv.$$

- R vanishes on $S^{1,0}_D(\mathcal{T})$.

Localization

- λ_z, $z \in \mathcal{N}$, form a partition of unity:

$$\langle R, v\rangle = \sum_{z\in\mathcal{N}} \langle R, \lambda_z v\rangle, \quad \|\nabla v\|^p_{L^p(\Omega)} = \sum_{z\in\mathcal{N}} \|\nabla v\|^p_{L^p_z(\omega_z)}$$

- $v_z \in \mathbb{R}$ arbitrary subject to the condition $\lambda_z v_z \in S^{1,0}_D(\mathcal{T})$:

$$\langle R, \lambda_z v\rangle = \langle R, \lambda_z(v - v_z)\rangle$$

- L^p-representation of R:

$$\langle R, \lambda_z(v - v_z)\rangle = \int_{\omega_z} r\lambda_z(v - v_z) + \int_{\sigma_z} j\lambda_z(v - v_z)$$

Basic Poincaré-type Inequalities

For every vertex $z \in \mathcal{N}$ there are constants $c_p(\omega_z)$, $c_p(\sigma_z)$ and $v_z \in \mathbb{R}$ with $\lambda_z v_z \in S^{1,0}_D(\mathcal{T})$ and:

$$\|v - v_z\|_{L^p_z(\omega_z)} \le c_p(\omega_z) h_z \|\nabla v\|_{L^p_z(\omega_z)}$$

$$\Big\{\sum_{E\subset\sigma_z} h_E^{\perp} \|v - v_z\|^p_{L^p_z(E)}\Big\}^{\frac{1}{p}} \le c_p(\sigma_z) h_z \|\nabla v\|_{L^p_z(\omega_z)}$$

with h_z the diameter of ω_z and $h_E^{\perp} = \mu_{d,z}(\omega_E)/\mu_{d-1,z}(E)$.

Vertex Oriented A Posteriori Error Estimate

Set

$$\eta_z = h_z\Big[c_p(\omega_z)\|r\|_{L^{p'}_z(\omega_z)} + c_p(\sigma_z)\Big\{\sum_{E\subset\sigma_z} (h_E^{\perp})^{1-p'} \|j\|^{p'}_{L^{p'}_z(E)}\Big\}^{\frac{1}{p'}}\Big]$$

then

$$\|R\|_{W^{-1,p'}(\Omega)} \le \Big\{\sum_{z\in\mathcal{N}} \eta_z^{p'}\Big\}^{\frac{1}{p'}}.$$

Element Oriented A Posteriori Error Estimate

Set

$$c_p(K) = \max_{z\in\mathcal{N}_K}\Big\{c_p(\omega_z)\frac{h_z}{h_K}\Big\},$$

$$c_p(E) = \max_{z\in\mathcal{N}_E}\Big\{c_p(\sigma_z)\frac{h_z}{h_E^{1-\frac{1}{p}}(h_E^{\perp})^{\frac{1}{p}}}\Big\}$$

then

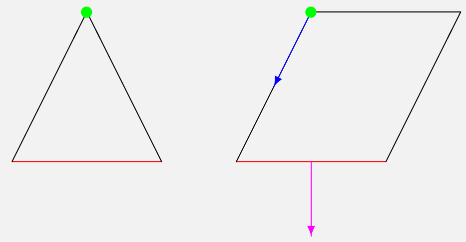
$$\|R\|_{W^{-1,p'}(\Omega)} \le \Big\{\sum_{K\in\mathcal{T}} c_p(K)^{p'} h_K^{p'} \|r\|^{p'}_{L^{p'}(K)}\Big\}^{\frac{1}{p'}} + \Big\{\sum_{E\in\mathcal{E}} c_p(E)^{p'} h_E \|j\|^{p'}_{L^{p'}(E)}\Big\}^{\frac{1}{p'}}.$$

Typical Values of Constants

	ω_z convex		ω_z not convex	
	$z \notin \Gamma_D$	$z \in \Gamma_D$	$z \notin \Gamma_D$	$z \in \Gamma_D$
$c_p(\omega_z)$	0.3	1.0	0.7	2.0
$c_p(\sigma_z)$	0.7	1.7	1.4	3.4
$c_p(K)$	0.7	2.2	1.4λ	4.1λ
$c_p(E)$	$2.4\sqrt{\lambda}$	$5.5\sqrt{\lambda}$	$5.8\sqrt{\lambda}$	$13.5\sqrt{\lambda}$

where λ measures the anisotropy

Vector Field



Vector Field

The vector field

$$\gamma_{K,E} = \begin{cases} x - a_{K,E} & \text{if } K \text{ is a simplex,} \\ \frac{(x-a_{K,E})\cdot n_{K,E}}{m_{K,E}\cdot n_{K,E}} m_{K,E} & \text{if } K \text{ is a parallelepiped,} \end{cases}$$

has the properties:

$$\operatorname{div} \gamma_{K,E} = \nu_K = \begin{cases} d & \text{if } K \text{ is a simplex,} \\ 1 & \text{if } K \text{ is a parallelepiped,} \end{cases}$$

$$\gamma_{K,E} \cdot n_K = 0 \qquad \text{on } \partial K \setminus E,$$

$$\gamma_{K,E} \cdot n_K = \frac{\nu_K \mu_d(K)}{\mu_{d-1}(E)} \qquad \text{on } E,$$

$$\|\gamma_{K,E}\|_{L^\infty(K)} \le h_K.$$

Trace Identity

The Gauß theorem applied to $w\gamma_{K,E}$ and $\lambda_z w \gamma_{K,E}$ yields:

$$\frac{1}{\mu_{d-1}(E)} \int_E w = \frac{1}{\mu_d(K)} \int_K w + \frac{1}{\nu_K \mu_d(K)} \int_K \gamma_{K,E} \cdot \nabla w,$$

$$\frac{1}{\mu_{d-1,z}(E)} \int_E \lambda_z w = \frac{1}{\mu_{d,z}(K)} \int_K \lambda_z w + \frac{1}{(\nu_K + 1)\mu_{d,z}(K)} \int_K \lambda_z \gamma_{K,E} \cdot \nabla w.$$

Trace Inequalities

The trace identity applied to $|u|^q$ yields:

$$\frac{1}{\mu_{d-1,z}(E)}\|u\|^q_{L^q_z(E)} \le \frac{1}{\mu_{d,z}(K)}\|u\|^q_{L^q_z(K)} + \frac{qh_K}{(\nu_K+1)\mu_{d,z}(K)}\|u\|^{q-1}_{L^q_z(K)}\|\nabla u\|_{L^q_z(K)}.$$

The above trace inequality applied to the elements in ω_z and their faces gives:

$$\sum_{E\subset\sigma_z} h_E^\perp\|u\|^q_{L^q_z(E)} \le d\|u\|^q_{L^q_z(\omega_z)} + d\max_{K\subset\omega_z}\frac{qh_K}{\nu_K+1}\|u\|^{q-1}_{L^q_z(\omega_z)}\|\nabla u\|_{L^q_z(\omega_z)}.$$

Contribution of the Skeleton

The trace inequality for σ_z implies:

$$c_p(\sigma_z) \le c_p(\omega_z)\Big\{d\big[1+\frac{p}{c_p(\omega_z)}\max_{K\subset\omega_z}\frac{h_K}{(\nu_K+1)h_z}\big]\Big\}^{\frac{1}{p}} \le \Big\{d\big[1+\frac{p}{2}\big]\Big\}^{\frac{1}{p}}\max\Big\{c_p(\omega_z), c_p(\omega_z)^{1-\frac{1}{p}}\Big\}.$$

Tasks

We must bound

- $c_p(\omega_z)$ for $z\in\Gamma_D$ (Friedrich's inequality),
- $c_p(\omega_z)$ for $z\notin\Gamma_D$ and convex ω_z,
- $c_p(\omega_z)$ for $z\notin\Gamma_D$ and non-convex ω_z.
- Set

$$\mathrm{C}_{P,p}(\omega_z) = \sup_{u\in W^{1,p}(\omega_z)}\inf_{c\in\mathbb{R}}\frac{\|u-c\|_{L^p_z(\omega_z)}}{h_z\|\nabla u\|_{L^p_z(\omega_z)}}.$$

Vertices on the Dirichlet Boundary

The trace inequality for elements and faces yields for all Dirichlet vertices

$$c_p(\omega_z) \le \Big[1+M_z\max_{E\subset\Gamma_{D,z}}\Big(\frac{h_z^\perp}{h_E^\perp}\Big)^{\frac{1}{p}}\Big]\mathrm{C}_{P,p}(\omega_z) + M_z\max_{E\subset\Gamma_{D,z}}\Big[\frac{h_{\omega_E}}{h_z(\nu_{\omega_E}+1)}\Big(\frac{h_z^\perp}{h_E^\perp}\Big)^{\frac{1}{p}}\Big]$$

where $M_z \le \max_{K\subset\omega_z}(d+1-\nu_K)$ is the maximal number of Dirichlet faces per element, $\Gamma_{D,z}=\Gamma_D\cap\partial\omega_z$ and $h_z^\perp = \mu_{d,z}(\omega_z)/\mu_{d-1,z}(\Gamma_{D,z})$.

Convex Domains

If ω_z is convex a result of Chua and Wheeden implies

$$\mathrm{C}_{P,p}(\omega_z) \le \overline{\mathrm{C}}_{P,p} \le \begin{cases} \frac{1}{2} & \text{for } p = 1, \\ \frac{1}{\pi} & \text{for } p = 2, \\ 2\left(\frac{p}{2}\right)^{\frac{1}{p}} & \text{for general } p. \end{cases}$$

Non-Convex Domains

If ω_z is not convex we have

$$\mathrm{C}_{P,p}(\omega_z) \le 4\overline{\mathrm{C}}_{P,p}(n_z - 1)^{1-\frac{1}{p}} \left[\frac{1}{2} + \frac{p}{2\overline{\mathrm{C}}_{P,p}}\right]^{\frac{1}{p}} \max_{1\le i\le n_z} \frac{h_{K_i}\mu_d(\omega_z)^{\frac{1}{p}}}{\mu_d(K_i)^{\frac{1}{p}} h_z}$$

and

$$\mathrm{C}_{P,p}(\omega_z) \le 12d\overline{\mathrm{C}}_{P,p}\kappa_z^{\frac{d}{p}-1}\left(\frac{1}{2} + \frac{1}{2}\max\left\{\kappa_z^{-d}K_{p,d}(\kappa_z), \frac{p}{d\overline{\mathrm{C}}_{P,p}}\right\}\right)^{\frac{1}{p}}$$

with $\kappa_z = \frac{3d}{2}\max_{y\in\partial\omega_z}|y-z| / \min_{y\in\partial\omega_z}|y-z|$ and

$$K_{p,d}(x) \le \begin{cases} \frac{1}{d}\left(\frac{p-1}{|d-p|}\right)^{p-1} x^{\max\{p,d\}} & \text{if } p \ne d, \\ \frac{1}{d}\left(\ln x\right)^{d-1} & \text{if } p = d. \end{cases}$$

Necessary Modifications

- Replace the primal norm by $|||v||| = \left\{\varepsilon\| \, |\nabla\cdot| \, \|^2_{L^2(\Omega)} + \|\cdot\|^2_{L^2(\Omega)}\right\}^{\frac{1}{2}}$.
- Replace the dual norm by $|||\cdot|||_* = \sup\{\langle\cdot, v\rangle : |||v||| = 1\}$.
- Replace h_z by $\min\left\{\varepsilon^{-\frac{1}{2}}h_z, 1\right\}$.
- Replace $c_2(\omega_z)$ by $\max\left\{c_2(\omega_z), 1\right\}$.