

A Review of Robust A Posteriori Error Estimates

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Outline

Introduction

Diffusion Dominated Problems

Reaction Dominated Problems

Convection Dominated Problems

Non-Stationary Problems

Outlook

Objectives of A Posteriori Error Estimation

- ▶ Derive upper error bounds (**reliability**)

$$\|e\| \leq c^* \eta.$$

- ▶ Derive lower error bounds (**efficiency**)

$$\eta \leq c_* \|e\|.$$

- ▶ $c_* c^*$ determines the quality of the error estimate.
- ▶ $c_* c^*$ should be uniformly bounded w.r.t. to any mesh-size or parameter of the pde (**robustness**).

Diffusion Dominated Model Problem

$$\begin{aligned} -\Delta u + u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \Omega \end{aligned}$$

Variational Formulation

Find $u \in X$ such that

$$a(u, v) = \langle \ell, v \rangle$$

holds for all $v \in Y$.

$$X = H_0^1(\Omega)$$

$$Y = H_0^1(\Omega)$$

$$a(u, v) = \int_{\Omega} \{ \nabla u \cdot \nabla v + uv \}$$

$$\langle \ell, v \rangle = \int_{\Omega} f v$$

Energy norm

$$\|u\|^2 = a(u, u)$$

Meshes

- ▶ **Affine equivalence:** \mathcal{T} consists of triangles or parallelograms, if $d = 2$, or tetrahedrons or parallelepipeds, if $d = 3$.
- ▶ **Admissibility:** Any two elements in \mathcal{T} are either disjoint or share a vertex or a complete edge or – if $d = 3$ – a complete face.
- ▶ **Shape regularity:** For any element K , the ratio of its diameter h_K to the diameter ρ_K of the largest ball inscribed into K is bounded independently of K .
- ▶ **Shape parameter:**

$$c_{\mathcal{T}} = \max_{K \in \mathcal{T}} \frac{h_K}{\rho_K}.$$

Finite Element Spaces

$$R_k(K) = \begin{cases} \text{span}\{x^\alpha : |\alpha|_1 \leq k\} & \text{if } K \text{ is a triangle or} \\ & \text{a tetrahedron,} \\ \text{span}\{x^\alpha : |\alpha|_\infty \leq k\} & \text{if } K \text{ is a parallelogram or} \\ & \text{a parallelepiped.} \end{cases}$$

$$S^{k,-1}(\mathcal{T}) = \{v : \Omega \rightarrow \mathbb{R} : v|_K \in R_k(K) \text{ for all } K \in \mathcal{T}\},$$

$$S^{k,0}(\mathcal{T}) = S^{k,-1}(\mathcal{T}) \cap C(\bar{\Omega}),$$

$$S_0^{k,0}(\mathcal{T}) = S^{k,0}(\mathcal{T}) \cap H_0^1(\Omega)$$

$$= \{v \in S^{k,0}(\mathcal{T}) : v = 0 \text{ on } \Gamma\}.$$

Discrete Problem

Find $u_{\mathcal{T}} \in S_0^{k,0}(\mathcal{T})$ such that

$$a(u_{\mathcal{T}}, v_{\mathcal{T}}) = \langle \ell, v_{\mathcal{T}} \rangle$$

holds for all $v_{\mathcal{T}} \in S_0^{k,0}(\mathcal{T})$.

Equivalence of Error and Residual

Residual $R(u_{\mathcal{T}})$

$$\begin{aligned}\langle R(u_{\mathcal{T}}), v \rangle &= \langle \ell, v \rangle - a(u_{\mathcal{T}}, v) \\ &= a(e, v)\end{aligned}$$

Continuity of a

$$\| \| R(u_{\mathcal{T}}) \| \|_* \leq \| \| e \| \|$$

Coercivity of a

$$\| \| e \| \| \leq \| \| R(u_{\mathcal{T}}) \| \|_*$$

Galerkin Orthogonality

$$\langle R(u_{\mathcal{T}}), v_{\mathcal{T}} \rangle = 0$$

for all $v_{\mathcal{T}} \in S_0^{k,0}(\mathcal{T})$.

L^2 -Representation of the Residual

Integration by parts element-wise

$$\begin{aligned}\langle R(u_{\mathcal{T}}), v \rangle &= \sum_{K \in \mathcal{T}} \int_K \underbrace{\{f + \Delta u_{\mathcal{T}} - u_{\mathcal{T}}\}}_{=R_K(u_{\mathcal{T}})} v \\ &+ \sum_{E \in \mathcal{E}_{\mathcal{T}}} \int_E \underbrace{-\mathbb{J}_E(n_E \cdot \nabla u_{\mathcal{T}})}_{=R_E(u_{\mathcal{T}})} v\end{aligned}$$

Quasi-Interpolation Operator

$$I_{\mathcal{T}}v = \sum_{x \in \mathcal{N}_{\mathcal{T}}} \lambda_x \pi_x v$$

with

$$\pi_x v = \frac{1}{\mu_d(\omega_x)} \int_{\omega_x} v$$

Local error estimates

$$\|v - I_{\mathcal{T}}v\|_{L^2(K)} \leq c_{A1} h_K \|\nabla v\|_{L^2(\tilde{\omega}_K)}$$

$$\|v - I_{\mathcal{T}}v\|_{L^2(E)} \leq c_{A2} h_E^{\frac{1}{2}} \|\nabla v\|_{L^2(\tilde{\omega}_E)}$$

Upper Bound

Galerkin orthogonality, L^2 -representation, Cauchy-Schwarz inequality, and local error estimates:

$$\begin{aligned} \|R(u_T)\|_* &\leq c^* \left\{ \sum_{K \in \mathcal{T}} h_K^2 \|R_K(u_T)\|_{L^2(K)}^2 \right. \\ &\quad \left. + \sum_{E \in \mathcal{E}_T} h_E \|R_E(u_T)\|_{L^2(E)}^2 \right\}^{\frac{1}{2}} \\ &= c^* \left\{ \sum_{K \in \mathcal{T}} \eta_K^2 \right\}^{\frac{1}{2}} \end{aligned}$$

c^* only depends on c_{A1} , c_{A2} which in turn only depend on the shape parameter c_T .

Local Cut-Off Functions

$$\psi_K = \alpha_K \prod_{x \in \mathcal{N}_K} \lambda_x$$

$$\psi_E = \beta_E \prod_{x \in \mathcal{N}_E} \lambda_x$$

Weights are determined by

$$\max_{x \in K} \psi_K(x) = 1$$

$$\max_{x \in E} \psi_E(x) = 1$$

Inverse Estimates

For all $v \in R_k(K)$, $\sigma \in R_k(E)$:

$$\begin{aligned} \|v\|_{L^2(K)} &\leq c_{I1,k} \|\psi_K^{\frac{1}{2}} v\|_{L^2(K)} \\ \|\nabla(\psi_K v)\|_{L^2(K)} &\leq c_{I2,k} h_K^{-1} \|v\|_{L^2(K)} \\ \|\sigma\|_{L^2(E)} &\leq c_{I3,k} \|\psi_E^{\frac{1}{2}} \sigma\|_{L^2(E)} \\ \|\nabla(\psi_E \sigma)\|_{L^2(K)} &\leq c_{I4,k} h_K^{-\frac{1}{2}} \|\sigma\|_{L^2(E)} \\ \|\psi_E \sigma\|_{L^2(K)} &\leq c_{I5,k} h_K^{\frac{1}{2}} \|\sigma\|_{L^2(E)} \end{aligned}$$

Lower Bound

Insert functions $\psi_K R_K(u_T)$ and $\psi_E R_E(u_T)$ as test-functions in L^2 -representation of residual:

$$\eta_K \leq c_* \|e\|_{\omega_K}$$

c_* only depends on $c_{I1,k}, \dots, c_{I5,k}$ which in turn only depend on the polynomial degree k and the shape parameter c_T .

Summary of Crucial Steps

- ▶ Equivalence of error and residual.
- ▶ Galerkin orthogonality.
- ▶ L^2 -representation of residual.
- ▶ Local error estimates for quasi-interpolation operator.
- ▶ Inverse estimates for local cut-off functions.

Reaction Dominated Model Problem

$$\begin{aligned} -\varepsilon \Delta u + u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \Omega \end{aligned}$$

with $0 < \varepsilon \ll 1$.

Change bilinear form a and energy norm correspondingly:

$$\|u\|^2 = \varepsilon \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2$$

Difficulties

- ▶ Choose standard H^1 -norm:
 - ▶ Norm of a is ≈ 1 .
 - ▶ Coercivity constant of a is $\approx \varepsilon$.
 - ▶ $c_* c^* \approx \varepsilon^{-1}$.
- ▶ Choose energy norm:
 - ▶ c_{A1}, c_{A2} depend on ε .
 - ▶ $c_{I1,k}, \dots, c_{I5,k}$ depend on ε .
 - ▶ $c_* c^* \approx \varepsilon^{-1}$.
- ▶ Need a new idea!

Improved Interpolation Error Estimates

Elements

$$\begin{aligned} \|v - I_{\mathcal{T}}v\|_{L^2(K)} &\leq c_{A1} h_K \|\nabla v\|_{L^2(\tilde{\omega}_K)} \\ &\leq c_{A1} \varepsilon^{-\frac{1}{2}} h_K \|v\|_{\tilde{\omega}_K} \\ \|v - I_{\mathcal{T}}v\|_{L^2(K)} &\leq c_{A3} \|v\|_{L^2(\tilde{\omega}_K)} \\ &\leq c_{A3} \|v\|_{\tilde{\omega}_K} \\ \|v - I_{\mathcal{T}}v\|_{L^2(K)} &\leq \tilde{c}_{A1} \underbrace{\min\{\varepsilon^{-\frac{1}{2}} h_K, 1\}}_{=\alpha_K} \|v\|_{\tilde{\omega}_K} \end{aligned}$$

Trace Inequality

Reference element \widehat{K} , $\widehat{E}_i = \widehat{K} \cap \{x_i = 0\}$, $\widehat{v} \in H^1(\widehat{K})$ vanishes on edge or face opposite to \widehat{E}_i :

$$\|\widehat{v}\|_{L^2(\widehat{E}_i)} \leq \left\{ 2\|\widehat{v}\|_{L^2(\widehat{K})} \left\| \frac{\partial \widehat{v}}{\partial x_i} \right\|_{L^2(\widehat{K})} \right\}^{\frac{1}{2}}$$

General element K and edge or face E , $v \in H^1(K)$:

$$\|v\|_{L^2(E)} \leq c \left\{ h_K^{-1} \|v\|_{L^2(K)}^2 + \|v\|_{L^2(K)} \|\nabla v\|_{L^2(K)} \right\}^{\frac{1}{2}}$$

c only depends on shape parameter $c_{\mathcal{T}}$ and number of vertices of E .

Improved Interpolation Error Estimates

Edges or Faces

$$\begin{aligned} \|v - I_{\mathcal{T}}v\|_{L^2(E)} &\leq c \left\{ h_K^{-1} \alpha_K^2 + \alpha_K \varepsilon^{-\frac{1}{2}} \right\}^{\frac{1}{2}} \|v\|_{\widetilde{\omega}_E} \\ &\leq \widetilde{c}_{A2} \varepsilon^{-\frac{1}{4}} \alpha_E^{\frac{1}{2}} \|v\|_{\widetilde{\omega}_E} \end{aligned}$$

Upper Bound

Galerkin orthogonality, L^2 -representation, Cauchy-Schwarz inequality, and improved interpolation error estimates:

$$\begin{aligned} \|R(u_{\mathcal{T}})\|_* &\leq c^* \left\{ \sum_{K \in \mathcal{T}} \alpha_K^2 \|R_K(u_{\mathcal{T}})\|_{L^2(K)}^2 \right. \\ &\quad \left. + \sum_{E \in \mathcal{E}_{\mathcal{T}}} \varepsilon^{-\frac{1}{2}} \alpha_E^{\frac{1}{2}} \|R_E(u_{\mathcal{T}})\|_{L^2(E)}^2 \right\}^{\frac{1}{2}} \\ &= c^* \left\{ \sum_{K \in \mathcal{T}} \eta_K^2 \right\}^{\frac{1}{2}} \end{aligned}$$

c^* only depends on \widetilde{c}_{A1} , \widetilde{c}_{A2} which in turn only depend on the shape parameter $c_{\mathcal{T}}$.

Inverse Estimates

Elements

$$\begin{aligned} \|\psi_K v\|_K &\leq \left\{ \varepsilon c_{I2,k}^2 h_K^{-2} + 1 \right\}^{\frac{1}{2}} \|v\|_{L^2(K)} \\ &\leq \widetilde{c}_{I2,k} \alpha_K^{-1} \|v\|_{L^2(K)} \end{aligned}$$

Inverse Estimates

Edges or Faces

$$\begin{aligned} \|\psi_E v\|_K &\leq \left\{ \varepsilon c_{I4,k}^2 h_E^{-1} + c_{I5,k} h_E \right\}^{\frac{1}{2}} \|v\|_{L^2(E)} \\ &\leq \tilde{c}_{I4,k} h_E^{\frac{1}{2}} \alpha_K^{-1} \|v\|_{L^2(E)} \\ \|\psi_E v\|_{L^2(K)} &\leq c_{I5,k} h_E^{\frac{1}{2}} \|v\|_{L^2(E)} \end{aligned}$$

but we need

$$\begin{aligned} \|\psi_E v\|_K &\leq c \varepsilon^{\frac{1}{4}} \alpha_E^{-\frac{1}{2}} \|v\|_{L^2(E)} \\ \|\psi_E v\|_{L^2(K)} &\leq c \varepsilon^{\frac{1}{4}} \alpha_E^{\frac{1}{2}} \|v\|_{L^2(E)} \end{aligned}$$

Modified Cut-Off Functions

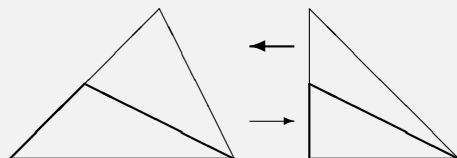
- ▶ Transform K, E to reference element \widehat{K} and edge or face \widehat{E}_d .
- ▶ Apply transformation

$$(x_1, \dots, x_{d-1}, x_d) \mapsto (x_1, \dots, x_{d-1}, \delta x_d).$$

- ▶ Transform back to K, E and obtain "squeezed" element K_δ .
- ▶ Denote by $\lambda_{x,\delta}$ the nodal shape functions of K_δ and set

$$\psi_{E,\delta} = \beta_E \prod_{x \in \mathcal{N}_E} \lambda_{x,\delta}.$$

Squeezing Elements



Inverse Estimates for Modified Cut-Off Functions

$$\begin{aligned} \|\psi_{E,\delta} v\|_{L^2(K)} &\leq c \delta^{\frac{1}{2}} h_E^{\frac{1}{2}} \|v\|_{L^2(E)} \\ \|\nabla(\psi_{E,\delta} v)\|_{L^2(K)} &\leq c \delta^{-\frac{1}{2}} h_E^{-\frac{1}{2}} \|v\|_{L^2(E)} \end{aligned}$$

choose

$$\begin{aligned} \delta &= \varepsilon^{\frac{1}{2}} h_E^{-1} \alpha_E \\ &= \min\{1, \varepsilon^{\frac{1}{2}} h_E^{-1}\} \\ \|\psi_{E,\delta} v\|_{L^2(K)} &\leq c \varepsilon^{\frac{1}{4}} \alpha_E^{\frac{1}{2}} \|v\|_{L^2(E)} \\ \|\psi_{E,\delta} v\|_K &\leq c \varepsilon^{\frac{1}{4}} \alpha_E^{-\frac{1}{2}} \|v\|_{L^2(E)} \end{aligned}$$

Lower Bound

Insert functions $\psi_K R_K(u_{\mathcal{T}})$ and $\psi_{E,\delta} R_E(u_{\mathcal{T}})$ as test-functions in L^2 -representation of residual:

$$\eta_K \leq c_* \|e\|_{\omega_K}$$

c_* only depends on the polynomial degree k and the shape parameter $c_{\mathcal{T}}$.

Summary of Crucial Steps

- ▶ Modified weights for element and edge or face residuals.
- ▶ Improved interpolation error estimates.
- ▶ Modified local cut-off functions.

Convection Dominated Model Problem

$$\begin{aligned} -\varepsilon \Delta u + \underline{a} \cdot \nabla u + u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \Omega \end{aligned}$$

with $0 < \varepsilon \ll 1$, $\underline{a} \in \mathbb{R}^d$ and $|\underline{a}| = 1$.

Change bilinear form a correspondingly; energy norm remains unchanged.

SUPG-Discretization

Find $u_{\mathcal{T}} \in S_0^{k,0}(\mathcal{T})$ such that

$$\begin{aligned} &a(u_{\mathcal{T}}, v_{\mathcal{T}}) \\ &+ \sum_{K \in \mathcal{T}} \delta_K \int_K \{-\varepsilon \Delta u_{\mathcal{T}} + \underline{a} \cdot \nabla u_{\mathcal{T}} + u_{\mathcal{T}}\} \underline{a} \cdot \nabla v_{\mathcal{T}} \\ &= \langle \ell, v_{\mathcal{T}} \rangle \\ &+ \sum_{K \in \mathcal{T}} \delta_K \int_K f \underline{a} \cdot \nabla v_{\mathcal{T}} \end{aligned}$$

holds for all $v_{\mathcal{T}} \in S_0^{k,0}(\mathcal{T})$.

Difficulties

- ▶ Missing Galerkin orthogonality due to stabilization terms.
 - ▶ Must bound the consistency error.
- ▶ a is coercive w.r.t. the energy norm, but

$$a(u, v) \leq \|u\| \|v\| + \|\underline{a} \cdot \nabla u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}.$$

- ▶ Error and residual are not fully equivalent

$$\begin{aligned} \|e\| &\leq \|R(u_{\mathcal{T}})\|_* \\ \|R(u_{\mathcal{T}})\|_* &\leq \left\{1 + \varepsilon^{-\frac{1}{2}}\right\} \|e\|. \end{aligned}$$

- ▶ More detailed analysis yields $c^* \approx 1$ and $c_* \approx 1 + \varepsilon^{-\frac{1}{2}}h$.

Estimation of the Consistency Error

$$\begin{aligned} \langle R(u_{\mathcal{T}}), v_{\mathcal{T}} \rangle &= \sum_{K \in \mathcal{T}} \delta_K \int_K \{f + \varepsilon \Delta u_{\mathcal{T}} - \underline{a} \cdot \nabla u_{\mathcal{T}} - u_{\mathcal{T}}\} \underline{a} \cdot \nabla v_{\mathcal{T}} \\ &= \sum_{K \in \mathcal{T}} \delta_K \int_K R_K(u_{\mathcal{T}}) \underline{a} \cdot \nabla v_{\mathcal{T}} \\ &\leq \sum_{K \in \mathcal{T}} \delta_K \|R_K(u_{\mathcal{T}})\|_{L^2(K)} \|\underline{a} \cdot \nabla v_{\mathcal{T}}\|_{L^2(K)} \\ &\leq \sum_{K \in \mathcal{T}} \delta_K h_K^{-1} \alpha_K \|R_K(u_{\mathcal{T}})\|_{L^2(K)} \|v_{\mathcal{T}}\|_K \end{aligned}$$

Assume that $\delta_K \leq ch_K$. This holds for all schemes used in practice.

Robust Bounds for the Residual

$$\begin{aligned} \|R(u_{\mathcal{T}})\|_* &\leq c^* \left\{ \sum_{K \in \mathcal{T}} \eta_K^2 \right\}^{\frac{1}{2}} \\ \left\{ \sum_{K \in \mathcal{T}} \eta_K^2 \right\}^{\frac{1}{2}} &\leq c_* \|R(u_{\mathcal{T}})\|_* \end{aligned}$$

$c_* c^*$ is independent of ε .

Need robust estimates of the error in terms of the residual.

Modified Norm

Equip $X = H_0^1(\Omega)$ with norm

$$\|u\|_X = \|u\| + \|\underline{a} \cdot \nabla u\|_*$$

and $Y = H_0^1(\Omega)$ with the energy norm

$$\|v\|_Y = \|v\|.$$

a is continuous with norm 1

$$a(u, v) \leq \|u\|_X \|v\|_Y.$$

a is uniformly stable

$$\inf_{u \in X} \sup_{v \in Y} \frac{a(u, v)}{\|u\|_X \|v\|_Y} \geq \frac{1}{3}.$$

Uniform Stability

Choose $\theta \in (0, 1)$ and $w_\theta \in Y$ with

$$\|w_\theta\| = 1$$

and

$$\int_{\Omega} \underline{a} \cdot \nabla u w_\theta \geq \theta \|\underline{a} \cdot \nabla u\|_*.$$

Set

$$v_\theta = u + \frac{1}{2} \|u\| w_\theta.$$

Uniform Stability

Then

$$\|v_\theta\| \leq \frac{3}{2} \|u\|$$

and

$$\begin{aligned} a(u, v_\theta) &= a(u, u) + \frac{1}{2} \|u\| a(u, w_\theta) \\ &\geq \|u\|^2 + \frac{1}{2} \|u\| a(u, w_\theta) \\ &\geq \|u\|^2 + \frac{1}{2} \|u\| \left\{ \theta \|\underline{a} \cdot \nabla u\|_* - \|u\| \right\} \\ &= \frac{1}{2} \|u\| \left\{ \|u\| + \theta \|\underline{a} \cdot \nabla u\|_* \right\} \end{aligned}$$

Robust Error Estimate

$$\begin{aligned} \|e\| + \|\underline{a} \cdot \nabla e\|_* &\leq c^* \left\{ \sum_{K \in \mathcal{T}} \eta_K^2 \right\}^{\frac{1}{2}} \\ \left\{ \sum_{K \in \mathcal{T}} \eta_K^2 \right\}^{\frac{1}{2}} &\leq c_* \left\{ \|e\| + \|\underline{a} \cdot \nabla e\|_* \right\} \end{aligned}$$

$c_* c^*$ only depends on the polynomial degree k and the shape parameter $c_{\mathcal{T}}$.

SUPG-Norm

Mesh-dependent SUPG-norm:

$$\|v\| + \|\underline{a} \cdot \nabla v\|_{\mathcal{T}}$$

with

$$\|\underline{a} \cdot \nabla v\|_{\mathcal{T}}^2 = \sum_{K \in \mathcal{T}} \delta_K \|\underline{a} \cdot \nabla v\|_{L^2(K)}^2.$$

Relation to modified norm:

$$\begin{aligned} \|\underline{a} \cdot \nabla v\|_* &\leq \max_{K \in \mathcal{T}} \delta_K^{-\frac{1}{2}} \|\underline{a} \cdot \nabla v\|_{\mathcal{T}} \\ \|\underline{a} \cdot \nabla v\|_{\mathcal{T}} &\leq \max_{K \in \mathcal{T}} \delta_K^{\frac{1}{2}} \alpha_K^{-1} \|\underline{a} \cdot \nabla v\|_*. \end{aligned}$$

Relation of both norms depends on mesh Péclet number.

Interpretation of Results

- ▶ Error estimator is robust w.r.t. the modified norm.
- ▶ Error estimator may over-estimate the energy norm.
- ▶ Over-estimation of the energy norm indicates an unresolved layer.

Summary of Crucial Steps

- ▶ Estimation of the consistency error.
- ▶ Problem-adapted norm for error estimation.

Non-Stationary Model Problem

$$\begin{aligned} \partial_t u - \varepsilon \Delta u + \underline{a} \cdot \nabla u + u &= f && \text{in } \Omega \times (0, T] \\ u &= 0 && \text{on } \Gamma \times (0, T] \\ u &= u_0 && \text{in } \Omega \end{aligned}$$

Space and Norm

$$\begin{aligned} X(a, b) &= \left\{ u \in L^2(a, b; H_0^1(\Omega)) \cap L^\infty(a, b; L^2(\Omega)) : \right. \\ &\quad \left. \partial_t u + \underline{a} \cdot \nabla u \in L^2(a, b; H^{-1}(\Omega)) \right\}, \\ \|u\|_{X(a,b)} &= \left\{ \text{ess. sup}_{a < t < b} \|u(\cdot, t)\|_{L^2(\Omega)}^2 + \int_a^b \| \|u(\cdot, t)\| \|^2 dt \right. \\ &\quad \left. + \int_a^b \| \|(\partial_t u + \underline{a} \cdot \nabla u)(\cdot, t)\| \|^2 dt \right\}^{\frac{1}{2}}. \end{aligned}$$

Space-Time Elements

Temporal partition:

$$\mathcal{I} = \{[t_{n-1}, t_n] : 1 \leq n \leq N_{\mathcal{I}}\}.$$

Time-steps:

$$\tau_n = t_n - t_{n-1}.$$

Spatial partitions:

$$\mathcal{T}_0, \dots, \mathcal{T}_{N_{\mathcal{I}}}.$$

Finite element spaces:

$$X_0, \dots, X_{N_{\mathcal{I}}}.$$

Conditions

- ▶ The partitions \mathcal{T}_n are **affine equivalent**, **admissible**, and **shape regular**.
- ▶ **Transition condition**: For every n there is an affine equivalent, admissible, and shape regular partition $\tilde{\mathcal{T}}_n$ such that it is a refinement of both \mathcal{T}_n and \mathcal{T}_{n-1} and such that

$$c_{\mathcal{I}} = \sup_{1 \leq n \leq N_{\mathcal{I}}} \sup_{K \in \tilde{\mathcal{T}}_n} \sup_{\substack{K' \in \mathcal{T}_n \\ K \subset K'}} \frac{h_{K'}}{h_K} < \infty$$

uniformly with respect to all partitions \mathcal{I} .

- ▶ **Degree condition**: Each X_n consists of continuous functions which are piecewise polynomials, the degrees being at least one and being bounded uniformly with respect to all partitions \mathcal{T}_n and \mathcal{I} .

Discrete Problem

Find $u_{\mathcal{T}_n}^n \in X_n$, $0 \leq n \leq N_{\mathcal{I}}$, such that

$$u_{\mathcal{T}_0}^0 = \pi_0 u_0$$

and, for $n = 1, \dots, N_{\mathcal{I}}$,

$$\begin{aligned} & \int_{\Omega} \frac{u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}}{\tau_n} v_{\mathcal{T}_n} + a(\theta \nabla u_{\mathcal{T}_n}^n + (1 - \theta) \nabla u_{\mathcal{T}_{n-1}}^{n-1}, v_{\mathcal{T}_n}) \\ & + \sum_{K \in \tilde{\mathcal{T}}_n} \delta_K \int_K \left(\frac{u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}}{\tau_n} + L(\theta u_{\mathcal{T}_n}^n + (1 - \theta) u_{\mathcal{T}_{n-1}}^{n-1}) \right) \underline{a} \cdot \nabla v_{\mathcal{T}_n} \\ & = \langle \ell, v_{\mathcal{T}_n} \rangle + \sum_{K \in \tilde{\mathcal{T}}_n} \delta_K \int_K f \underline{a} \cdot \nabla v_{\mathcal{T}_n} \end{aligned}$$

for all $v_{\mathcal{T}_n} \in X_n$.

Notation

$u_{\mathcal{I}}$ is piecewise affine w.r.t. time and equals $u_{\mathcal{T}_n}^n$ at time t_n :

$$\begin{aligned} u_{\mathcal{I}}(\cdot, t) &= \frac{1}{\tau_n} \left((t_n - t) u_{\mathcal{T}_{n-1}}^{n-1} + (t - t_{n-1}) u_{\mathcal{T}_n}^n \right) && \text{on } [t_{n-1}, t_n] \\ \partial_t u_{\mathcal{I}} &= \frac{1}{\tau_n} (u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}) && \text{on } [t_{n-1}, t_n]. \end{aligned}$$

Residual

Residual $R(u_{\mathcal{I}})$:

$$\begin{aligned}\langle R(u_{\mathcal{I}}), v \rangle &= \langle \ell, v \rangle - \int_{\Omega} \partial_t u_{\mathcal{I}} v - a(u_{\mathcal{I}}, v) \\ &= \int_{\Omega} \partial_t e v + a(e, v).\end{aligned}$$

Equivalence of Error and Residual

Insert any function $w \in L^2(0, T; H_0^1(\Omega))$ into the residual equation:

$$\|R(u_{\mathcal{I}})\|_{L^2(0, T; H^{-1}(\Omega))} \leq \sqrt{2} \|e\|_{X(0, T)}.$$

Insert error e into residual equation (**parabolic energy estimate**):

$$\begin{aligned}\|e\|_{X(0, t_n)} &\leq \left\{ 4 \|u_0 - \pi_0 u_0\|_{L^2(\Omega)}^2 \right. \\ &\quad \left. + 6 \|R(u_{\mathcal{I}})\|_{L^2(0, t_n; H^{-1}(\Omega))}^2 \right\}^{\frac{1}{2}}.\end{aligned}$$

Decomposition of the Residual

Temporal residual $R_{\tau}(u_{\mathcal{I}})$:

$$\langle R_{\tau}(u_{\mathcal{I}}), v \rangle = a([\theta u_{\mathcal{I}_n}^n + (1 - \theta) u_{\mathcal{I}_{n-1}}^{n-1} - u_{\mathcal{I}}], v).$$

Spatial residual $R_h(u_{\mathcal{I}})$:

$$\begin{aligned}\langle R_h(u_{\mathcal{I}}), v \rangle &= \langle \ell, v \rangle - \int_{\Omega} \partial_t u_{\mathcal{I}} v \\ &\quad - a([\theta \nabla u_{\mathcal{I}_n}^n + (1 - \theta) \nabla u_{\mathcal{I}_{n-1}}^{n-1}], v).\end{aligned}$$

Decomposition:

$$R(u_{\mathcal{I}}) = R_{\tau}(u_{\mathcal{I}}) + R_h(u_{\mathcal{I}}).$$

Spatial Residual

Spatial error indicator $\eta_{\mathcal{I}_n}^n$:

$$\eta_{\mathcal{I}_n}^n = \left\{ \sum_{K \in \tilde{\mathcal{T}}_n} \alpha_K^2 \|R_K\|_{L^2(K)}^2 + \sum_{E \in \mathcal{E}_{\tilde{\mathcal{T}}_n}} \varepsilon^{-\frac{1}{2}} \alpha_E \|R_E\|_{L^2(E)}^2 \right\}^{\frac{1}{2}}.$$

Standard arguments for stationary problems yield:

$$\begin{aligned}\|R_h(u_{\mathcal{I}})\|_* &\leq c^\dagger \eta_{\mathcal{I}_n}^n, \\ \eta_{\mathcal{I}_n}^n &\leq c_\dagger \|R_h(u_{\mathcal{I}})\|_*.\end{aligned}$$

$c^\dagger c_\dagger$ only depend on the polynomial degrees and on the shape parameters $c_{\tilde{\mathcal{T}}_n}$ and $c_{\mathcal{I}}$.

Temporal Residual

Representation of the temporal residual, the stability and continuity of the bilinear form a , and integration on $[t_{n-1}, t_n]$ yield:

$$\begin{aligned} & \left\{ \int_{t_{n-1}}^{t_n} \|R_\tau(u_{\mathcal{I}})(\cdot, s)\|_*^2 ds \right\}^{\frac{1}{2}} \\ & \leq \sqrt{\frac{2}{3}} \tau_n^{\frac{1}{2}} \left\{ \|u_{\mathcal{I}_n}^n - u_{\mathcal{I}_{n-1}}^{n-1}\|^2 + \|a \cdot \nabla(u_{\mathcal{I}_n}^n - u_{\mathcal{I}_{n-1}}^{n-1})\|_*^2 \right\}^{\frac{1}{2}}, \\ & \left\{ \int_{t_{n-1}}^{t_n} \|R_\tau(u_{\mathcal{I}})(\cdot, s)\|_*^2 ds \right\}^{\frac{1}{2}} \\ & \geq \frac{1}{24} \sqrt{\frac{3}{8}} \tau_n^{\frac{1}{2}} \left\{ \|u_{\mathcal{I}_n}^n - u_{\mathcal{I}_{n-1}}^{n-1}\|^2 + \|a \cdot \nabla(u_{\mathcal{I}_n}^n - u_{\mathcal{I}_{n-1}}^{n-1})\|_*^2 \right\}^{\frac{1}{2}}. \end{aligned}$$

Preliminary Error Estimator

Define a preliminary space-time error estimator by:

$$\begin{aligned} \hat{\eta}_n &= \tau_n^{\frac{1}{2}} \left[\left(\eta_{\mathcal{I}_n}^n \right)^2 \right. \\ & \quad \left. + \|u_{\mathcal{I}_n}^n - u_{\mathcal{I}_{n-1}}^{n-1}\|^2 + \|a \cdot \nabla(u_{\mathcal{I}_n}^n - u_{\mathcal{I}_{n-1}}^{n-1})\|_*^2 \right]^{\frac{1}{2}}. \end{aligned}$$

Upper Error Bound

Equivalence of error and residual, the decomposition of the residual, the triangle inequality and the upper bounds for the spatial and temporal residuals yield:

$$\|e\|_{X(0,T)} \leq \tilde{c}^* \left\{ \|u_0 - \pi_0 u_0\|_{L^2(\Omega)}^2 + \sum_{n=1}^{N_{\mathcal{I}}} \hat{\eta}_n^2 \right\}^{\frac{1}{2}}.$$

Lower Error Bound

Equivalence of error and residual, the decomposition of the residual, the triangle inequality, the lower bounds for the spatial and temporal residuals, and a test-function of the form

$$w_{n,\tau} + (\alpha + 1) \left(\frac{t - t_{n-1}}{\tau_n} \right)^\alpha w_{n,h}$$

with a suitable α yield:

$$\hat{\eta}_n \leq \hat{c}_* \|e\|_{X(t_{n-1}, t_n)}.$$

$\hat{c}_* \hat{c}^*$ only depends on the polynomial degrees, the shape parameters $c_{\tilde{\mathcal{I}}_n}$ and the constant in the transition condition.

Difficulty

- ▶ Term $\|\underline{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1})\|_*$ in η_n is not computable.
- ▶ Inverse estimate spoils the robustness.
- ▶ Idea:
 - ▶ $\|\cdot\|_*$ is the dual norm of a reaction diffusion problem.
 - ▶ Solve an auxiliary discrete reaction diffusion problem.
 - ▶ Replace $\|\underline{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1})\|_*$ by the energy norm of the solution of the auxiliary problem and a residual error estimator for the auxiliary problem.

Auxiliary Problem

Find $\tilde{u}_{\mathcal{T}_n}^n \in S_0^{1,0}(\tilde{\mathcal{T}}_n)$ such that

$$\int_{\Omega} \{\varepsilon \nabla \tilde{u}_{\mathcal{T}_n}^n \cdot \nabla v_{\mathcal{T}_n} + \tilde{u}_{\mathcal{T}_n}^n v_{\mathcal{T}_n}\} = \int_{\Omega} \underline{a} \cdot \nabla (u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}) v_{\mathcal{T}_n}$$

holds for all $v_{\mathcal{T}_n} \in S_0^{1,0}(\tilde{\mathcal{T}}_n)$.

Estimation of Dual Norm

Set

$$\begin{aligned} \tilde{\eta}_{\mathcal{T}_n}^n = & \left\{ \sum_{K \in \tilde{\mathcal{T}}_n} \alpha_K^2 \|\underline{a} \cdot \nabla (u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}) + \varepsilon \Delta \tilde{u}_{\mathcal{T}_n}^n - \tilde{u}_{\mathcal{T}_n}^n\|_{L^2(K)}^2 \right. \\ & \left. + \sum_{E \in \mathcal{E}_{\tilde{\mathcal{T}}_n}} \varepsilon^{-\frac{1}{2}} \alpha_E \|\mathbb{J}_E(n_E \cdot \nabla \tilde{u}_{\mathcal{T}_n}^n)\|_{L^2(E)}^2 \right\}^{\frac{1}{2}}. \end{aligned}$$

Then

$$\begin{aligned} \tilde{c}_{\dagger} \left\{ \|\tilde{u}_{\mathcal{T}_n}^n\| + \tilde{\eta}_{\mathcal{T}_n}^n \right\} & \leq \|\underline{a} \cdot \nabla (u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1})\|_* \\ & \leq \tilde{c}_{\dagger} \left\{ \|\tilde{u}_{\mathcal{T}_n}^n\| + \tilde{\eta}_{\mathcal{T}_n}^n \right\}. \end{aligned}$$

Final Error Estimate

Define the final space-time error estimator by:

$$\begin{aligned} \eta_n = & \tau_n^{\frac{1}{2}} \left[\left(\eta_{\mathcal{T}_n}^n \right)^2 \right. \\ & \left. + \|u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}\|^2 + \|\tilde{u}_{\mathcal{T}_n}^n\|^2 + \left(\tilde{\eta}_{\mathcal{T}_n}^n \right)^2 \right]^{\frac{1}{2}}. \end{aligned}$$

Then

$$\|e\|_{X(0,T)} \leq c_* \left\{ \|u_0 - \pi_0 u_0\|_{L^2(\Omega)}^2 + \sum_{n=1}^{N_{\mathcal{I}}} \eta_n^2 \right\}^{\frac{1}{2}},$$

$$\eta_n \leq c_* \|e\|_{X(t_{n-1}, t_n)}.$$

$c_* c^*$ only depends on the polynomial degrees and the shape parameters $c_{\tilde{\mathcal{T}}_n}$ and $c_{\mathcal{I}}$.

Summary of Crucial Steps

- ▶ Parabolic energy estimate for the equivalence of error and residual.
- ▶ Decomposition of the residual into a spatial and a temporal contribution.
- ▶ Standard stationary results for the robust estimation of the spatial residual.
- ▶ Robust estimation of the temporal residual.
- ▶ Judicious choice of test functions for the estimation of the combined residuals.
- ▶ Auxiliary discrete reaction diffusion problem and corresponding residual error estimate for handling the dual energy norm in the error estimator.

61/ 63

Extensions

- ▶ Variable coefficients and general source terms introduce additional data errors.
- ▶ General convection and reaction terms require





$$-\frac{1}{2} \operatorname{div} \underline{a} + b \geq \beta \geq 0$$

$$\|b\|_{L^\infty(\Omega)} \leq c_b \beta.$$

- ▶ Anisotropic meshes are handled by using suitable geometric quantities.
- ▶ Discontinuities of the diffusion should be caught by the coarsest mesh.
- ▶ Anisotropic diffusion can be handled by measuring the length in a diffusion-dependent norm.
- ▶ Non-linear problems are handled via the implicit function theorem.

62/ 63

References

-  [Robust a posteriori error estimators for a singularly perturbed reaction-diffusion equation](#),
Numer. Math. **78** (1998), no. 3, 479–493.
-  [Adaptive finite element methods for elliptic equations with non-smooth coefficients](#),
Numer. Math. **85** (2000), no. 4, 579–608.
-  [Robust a posteriori error estimates for stationary convection-diffusion equations](#),
SIAM J. Numer. Anal. **43** (2005), no. 4, 1766–1782
-  [Robust a posteriori error estimates for nonstationary convection-diffusion equations](#),
SIAM J. Numer. Anal. **43** (2005), no. 4, 1783–1802

63/ 63