Robust A Posteriori Error Estimates for Stabilized Finite Element Discretizations of Non-Stationary Convection-Diffusion Problems

L. Tobiska and R. Verfürth

Universität Magdeburg

Ruhr-Universität Bochum
www.ruhr-uni-bochum.de/num1

Oberwolfach / January 9, 2017

Goal

Establish residual a posteriori error estimates for stabilized finite element discretizations of non-stationary convection-diffusion problems which yield upper and lower bounds for the energy norm of the error that are uniform with respect to all possible relative sizes of convection to diffusion using a common framework for various stabilization methods.

Outline

Differential Equation

$$\begin{aligned} \partial_t u - \varepsilon\Delta u + \mathbf{a}\cdot\nabla u + \beta u &= f && \text{in } \Omega\times(0,T] \\ u &= 0 && \text{on } \Gamma\times(0,T] \\ u &= u_0 && \text{in } \Omega \end{aligned}$$

- $0 < \varepsilon \ll 1$, $\beta \ge 0$, $\mathbf{a}\in\mathbb{R}^d$, $|\mathbf{a}| \le 1$
- Results hold for variable coefficients and mixed boundary conditions. Then ε is a lower bound for the smallest eigenvalue of the diffusion and β is a lower bound for $b - \frac{1}{2}\operatorname{div}\mathbf{a}$ with b denoting the reaction.

Variational Problem

Find $u \in L^2(0,T;H_0^1(\Omega))$ with $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$ such that $u = u_0$ in L^2 and for all $t \in (0,T)$ and all $v \in H_0^1(\Omega)$

$$\langle \partial_t u \, , \, v \rangle + \underbrace{\int_\Omega \{\varepsilon \nabla u \cdot \nabla v + \mathbf{a} \cdot \nabla u v + \beta u v\}}_{=B(u,v)} = \underbrace{\int_\Omega f v}_{=\langle \ell \, , \, v \rangle}$$

Norms

- Energy norm $|||v||| = \left\{ \varepsilon \left\| \nabla v \right\|^2 + \beta \left\| v \right\|^2 \right\}^{\frac{1}{2}}$
- Dual norm $\quad |||\varphi|||_* = \sup\limits_{v \in H_0^1(\Omega) \setminus \{0\}} \dfrac{\langle \varphi, v \rangle}{|||v|||}$
- Error norm

$$\|u\|_{X(a,b)} = \left\{ \operatorname*{ess.\,sup}_{t \in (a,b)} \|u(\cdot,t)\|^2 + \int_a^b |||u(\cdot,t)|||^2 \, dt + \int_a^b |||(\partial_t u + \mathbf{a} \cdot \nabla u)(\cdot,t)|||_*^2 \, dt \right\}^{\frac{1}{2}}$$

Meshes and Spaces

- $\mathcal{I} = \{(t_{n-1}, t_n] : 1 \le n \le N_{\mathcal{I}}\}$ partition of $[0,T]$.
- $\tau_n = t_n - t_{n-1}$.
- $\mathcal{T}_n$, $0 \le n \le N_{\mathcal{I}}$, affine equivalent, admissible, shape regular partitions of Ω.
- Transition condition: There is a common refinement $\widetilde{\mathcal{T}}_n$ of $\mathcal{T}_n$ and $\mathcal{T}_{n-1}$ such that $h_K \le c h_{K'}$ for all $K \in \mathcal{T}_n$ and all $K' \in \widetilde{\mathcal{T}}_n$ with $K' \subset K$.
- $X_n \subset H_0^1(\Omega)$ finite element space corresponding to $\mathcal{T}_n$.

Discrete Problem

Find $u_{\mathcal{T}_n}^n \in X_n$, $0 \le n \le N_{\mathcal{I}}$, such that $u_{\mathcal{T}_0}^0 = \pi_0 u_0$ and, for $n = 1, \ldots, N_{\mathcal{I}}$ and all $v_{\mathcal{T}_n} \in X_n$ with $U^{n\theta} = \theta \nabla u_{\mathcal{T}_n}^n + (1-\theta) \nabla u_{\mathcal{T}_{n-1}}^{n-1}$

$$\left(\frac{u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}}{\tau_n} \, , \, v_{\mathcal{T}_n} \right) + B(U^{n\theta}, v_{\mathcal{T}_n}) + S_n(U^{n\theta}, v_{\mathcal{T}_n}) = \langle \ell \, , \, v_{\mathcal{T}_n} \rangle$$

- The stabilization term S_n is supposed to be linear in its second argument and affine in its first argument, it may depend on $\mathcal{T}_n$ and on f.
- Solution $u_{\mathcal{I}}$ is continuous piece-wise affine and equals $u_{\mathcal{T}_n}^n$ at t_n.

Stabilizations I

- Streamline diffusion method

$$S_n(u,v) = \sum_{K\in\mathcal{T}_n} \vartheta_K \int_K \left\{ \frac{1}{\tau_n}(u^n - u^{n-1}) - \varepsilon\Delta u \right.$$
$$\left. + \mathbf{a}\cdot\nabla u + \beta u - f \right\} \mathbf{a}\cdot\nabla v$$

with $\vartheta_K\,|\mathbf{a}| \le c h_K$

- Continuous interior penalty method

$$S_n(u,v) = \sum_{E\in\mathcal{E}_n} \vartheta_E \int_E \mathbb{J}_E(\mathbf{a}\cdot\nabla u)\mathbb{J}_E(\mathbf{a}\cdot\nabla v)$$

with $\vartheta_E \le c h_E^2$

Stabilizations II

- Local projection scheme

$$S_n(u,v) = \sum_{M\in\mathcal{M}_n} \vartheta_M \int_M \kappa_{\mathcal{M}}\left(\mathbf{a}\cdot\nabla u\right)\kappa_{\mathcal{M}}\left(\mathbf{a}\cdot\nabla v\right)$$

with $\vartheta_M\,|\mathbf{a}| \le c h_M$ or

$$S_n(u,v) = \sum_{M\in\mathcal{M}_n} \vartheta_M \int_M \kappa_{\mathcal{M}}\left(\nabla u\right)\kappa_{\mathcal{M}}\left(\nabla v\right)$$

with $\vartheta_M \le c\,|a|\,h_M$ and $\kappa_{\mathcal{M}}$ the orthogonal projection onto the complement of a suitable space of discontinuous functions on a macro-partition $\mathcal{M}$

Stabilizations III

- Subgrid scale approach

$$S_n(u,v) = \sum_{K\in\mathcal{T}_n} \vartheta_K \int_K \mathbf{a}\cdot\nabla\Pi_n(u)\,\mathbf{a}\cdot\nabla\Pi_n(v)$$

with $\vartheta_K\,|\mathbf{a}| \le c h_K$ or

$$S_n(u,v) = \sum_{K\in\mathcal{M}_n} \vartheta_K \int_K \nabla\Pi_n(u)\cdot\nabla\Pi_n(v)$$

with $\vartheta_K \le c\,|\mathbf{a}|\,h_K$ and Π_n the orthogonal projection onto a suitable space of unresolvable scales $Y_n \subset X_n$

Basic Steps

- Error and residual are equivalent.
- The residual splits into a spatial and a temporal residual.
- The norm of the sum of these is equivalent to the sum of their norms.
- Derive a reliable, efficient and robust error indicator for the temporal residual.
- Derive a reliable, efficient and robust error indicator for the spatial residual.
- All stabilizations yield the same spatial error indicator.

Equivalence of Error and Residual

- $u_{\mathcal{I}}$ is continuous piece-wise affine and equals $u^n_{\mathcal{T}_n}$ at t_n.
- Residual:

  $\langle R(u_{\mathcal{I}}) \,,\, v\rangle = \langle \ell \,,\, v\rangle - \langle \partial_t u_{\mathcal{I}} \,,\, v\rangle - B(u_{\mathcal{I}}, v)$
- Lower error-bound:

  $\|R(u_{\mathcal{I}})\|_{L^2(t_{n-1},t_n;H^{-1})} \leq \sqrt{2}\,\|u - u_{\mathcal{I}}\|_{X(t_{n-1},t_n)}$
- Upper error-bound:

  $\|u - u_{\mathcal{I}}\|_{X(0,t_n)} \leq \left\{4\,\|u_0 - \pi_0 u_0\|^2 + 6\,\|R(u_{\mathcal{I}})\|^2_{L^2(0,t_n;H^{-1})}\right\}^{\frac{1}{2}}$

Proof of the Equivalence

- Relation of residual and error:

  $\langle R(u_{\mathcal{I}}) \,,\, v\rangle = \langle \partial_t e \,,\, v\rangle + B(e, v)$
- Lower error-bound: Definition of primal and dual norm plus Cauchy-Schwarz inequality.
- Upper error-bound: Parabolic energy estimate with $v = e$ as test-function.

Decomposition of the Residual

- Recall $U^{n\theta} = \theta \nabla u^n_{\mathcal{T}_n} + (1-\theta)\nabla u^{n-1}_{\mathcal{T}_{n-1}}$
- Temporal residual:

  $\langle R_\tau(u_{\mathcal{I}}) \,,\, v\rangle = B(U^{n\theta} - u_{\mathcal{I}}, v)$
- Spatial residual:

  $\langle R_h(u_{\mathcal{I}}) \,,\, v\rangle = \langle \ell \,,\, v\rangle - \langle \partial_t u_{\mathcal{I}} \,,\, v\rangle - B(U^{n\theta}, v)$
- Splitting: $R(u_{\mathcal{I}}) = R_\tau(u_{\mathcal{I}}) + R_h(u_{\mathcal{I}})$
- Estimate for $L^2(t_{n-1}, t_n; H^{-1})$-norms:

$$\begin{aligned}\frac{1}{13}\left\{\|R_\tau(u_{\mathcal{I}})\|^2 + \|R_h(u_{\mathcal{I}})\|^2\right\}^{\frac{1}{2}} &\leq \|R_\tau(u_{\mathcal{I}}) + R_h(u_{\mathcal{I}})\| \\ &\leq \|R_\tau(u_{\mathcal{I}})\| + \|R_h(u_{\mathcal{I}})\|\end{aligned}$$

Motivation of the Lower Bound

- Strengthened Cauchy-Schwarz inequality for $v = c$ and $w = \frac{b-t}{b-a}$:

  $$\int_a^b vw = \frac{1}{2}c(b-a) = \frac{\sqrt{3}}{2}\,\|v\|_{(a,b)}\,\|w\|_{(a,b)}$$
- Hence:

  $$\|v + w\|^2_{(a,b)} \geq \left(1 - \frac{\sqrt{3}}{2}\right)\left\{\|v\|^2_{(a,b)} + \|w\|^2_{(a,b)}\right\}$$

Proof of the Lower Bound

- $R_h(u_{\mathcal{I}})$ is piece-wise constant.
- $R_\tau(u_{\mathcal{I}})$ is piece-wise affine: $R_\tau(u_{\mathcal{I}}) = \left(\theta - \frac{t-t_{n-1}}{\tau_n}\right) \rho^n$ with
  $\langle \rho^n \,, v\rangle = B(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}, v)$.
- Choose $v, w \in H^1_0(\Omega)$ such that
  $|||v||| = |||R_h(u_{\mathcal{I}})|||_*\,,\quad \langle R_h(u_{\mathcal{I}})\,, v\rangle = |||R_h(u_{\mathcal{I}})|||^2_*\,,$
  $|||w||| = |||\rho^n|||_*\,,\qquad \langle \rho^n \,, w\rangle = |||\rho^n|||^2_*.$
- Insert $3\left(\frac{t-t_{n-1}}{\tau_n}\right)^2 v + \frac{t_n - t}{\tau_n} w$ as test-function in representation of $R(u_{\mathcal{I}})$.

Estimation of the Temporal Residual

- $R_\tau(u_{\mathcal{I}}) = \left(\theta - \frac{t-t_{n-1}}{\tau_n}\right) \rho^n$ with
  $\langle \rho^n \,, v\rangle = B(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}, v)$.
- Upper bound:
  $|||\rho^n|||_* \le \left|\left|\left| u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}} \right|\right|\right| + \left|\left|\left| \mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}) \right|\right|\right|_*$
- Follows from definition of ρ^n and $|||\cdot|||_*$.
- Lower bound:
  $\frac{1}{3}\left\{ \left|\left|\left| u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}} \right|\right|\right| + \left|\left|\left| \mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}) \right|\right|\right|_* \right\} \le |||\rho^n|||_*$

Proof of the Lower Bound

- Set $w^n = u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}$ and choose $v \in H^1_0(\Omega)$ with $|||v||| = |||\mathbf{a}\cdot\nabla w^n|||_*$ and $(\mathbf{a}\cdot\nabla w^n \,, v) = |||\mathbf{a}\cdot\nabla w^n|||^2_*$
- Insert $\frac{1}{2}w^n + \frac{1}{2}v$ in the definition of ρ^n:

$$\left\langle \rho^n \,, \frac{1}{2}w^n + \frac{1}{2}v \right\rangle$$
$$= \underbrace{\frac{1}{2}(\varepsilon\nabla w^n \,, \nabla w^n) + \frac{1}{2}(\beta w^n \,, w^n)}_{=\frac{1}{2}\|w^n\|^2} + \underbrace{\frac{1}{2}(\mathbf{a}\cdot\nabla w^n, w^n)}_{=0}$$
$$+ \underbrace{\frac{1}{2}(\varepsilon\nabla w^n \,, \nabla v) + \frac{1}{2}(\beta w^n \,, v)}_{\ge -\frac{1}{2}\|w^n\|\|\mathbf{a}\cdot\nabla w^n\|_*} + \underbrace{\frac{1}{2}(\mathbf{a}\cdot\nabla w^n \,, v)}_{=\frac{1}{2}\|\mathbf{a}\cdot\nabla w^n\|^2_*}$$

Estimation of the Convective Derivative I

- Assume that $|\mathbf{a}| \le c_c\varepsilon$.
- Friedrichs' inequality implies
  $\left(\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})\,, v\right) \le |\mathbf{a}| \left\|\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})\right\| c_\Omega \|\nabla v\|.$
- Hence $\left|\left|\left| \mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}) \right|\right|\right|_* \le c_c c_\Omega \left|\left|\left| u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}} \right|\right|\right|$ and $\left|\left|\left| \mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}) \right|\right|\right|_*$ is equivalent to $\left|\left|\left| u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}} \right|\right|\right|$.

Estimation of the Convective Derivative II

- Assume that $|\mathbf{a}| \gg \varepsilon$.
- Auxiliary problem with analytical and discrete solutions Φ and $\Phi_{\mathcal{T}_n}$:

$$\varepsilon\,(\nabla\varphi\,,\,\nabla\psi) + \beta\,(\varphi\,,\,\psi) = \left(\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})\,,\,\psi\right) \qquad (*)$$

- $$\frac{1}{3}\left\{|||\Phi_{\mathcal{T}_n}||| + |||\Phi - \Phi_{\mathcal{T}_n}|||\right\} \le \left|\left|\left|\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})\right|\right|\right|_* \le |||\Phi_{\mathcal{T}_n}||| + |||\Phi - \Phi_{\mathcal{T}_n}|||$$
- $|||\Phi - \Phi_{\mathcal{T}_n}|||$ is equivalent to robust residual error indicator $\eta^n_{\mathcal{T}}$ for $(*)$.
- Hence $\left|\left|\left|\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})\right|\right|\right|_*$ is equivalent to $|||\Phi_{\mathcal{T}_n}||| + \eta^n_{\mathcal{T}}$.

Estimation of the Spatial Residual

- Spatial error indicator η^n_h:

$$\eta^n_h = \left\{\sum_{K\in\widetilde{\mathcal{T}}_n} {\alpha_K}^2\,\|R_K\|^2_K + \sum_{E\in\mathcal{E}_{\widetilde{\mathcal{T}}_n}} \varepsilon^{-\frac{1}{2}}\alpha_E\,\|R_E\|^2_E\right\}^{\frac{1}{2}}$$

- $\alpha_S = \min\left\{\varepsilon^{-\frac{1}{2}}h_S, \beta^{-\frac{1}{2}}\right\}$
- R_K and R_E are the usual element and interface residuals.
- Standard arguments for stationary problems yield:

$$|||R_h(u_{\mathcal{I}})|||_* \le c^\dagger\eta^n_h + |||I^*_{\mathcal{M}}R_h(u_{\mathcal{I}})|||_*,\ \eta^n_h \le c_\dagger\,|||R_h(u_{\mathcal{I}})|||_*.$$

- $|||I^*_{\mathcal{M}}R_h(u_{\mathcal{I}})|||_*$ measures the consistency error of the stabilization.
- $c_\dagger$, $c^\dagger$ only depend on the polynomial degrees and on the shape parameters of the partitions $\widetilde{\mathcal{T}}_n$.

Proof of the Upper Bound

- L^2-representation:

$$\langle R_h(u_{\mathcal{I}})\,,\,v\rangle = \int_\Omega rv + \int_\Sigma jv$$

- Quasi-interpolation error estimate:

$$\|v - I_{\mathcal{M}}v\|_K \le c\alpha_K\,|||v|||_{\widetilde{\omega}_K}$$

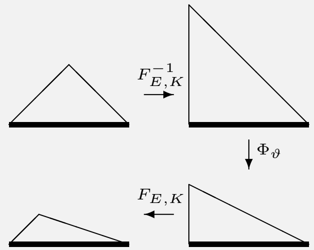
- Trace inequality:

$$\|v\|^2_E \le \frac{|E|}{|K|}\,\|v\|^2_K + \frac{2h_K\,|E|}{|K|}\,\|v\|_K\,\|\nabla v\|_K$$

Proof of the Lower Bound

- Insert $\psi_K R_K$ in L^2-representation with standard element cut-off functions ψ_K.
- Insert $\psi_{E,\vartheta}R_E$ in L^2-representation with squeezed face cut-off functions $\psi_{E,\vartheta}$ and

$$\vartheta = \varepsilon^{\frac{1}{2}}h_E^{-1}\alpha_E = \min\left\{1, \varepsilon^{\frac{1}{2}}h_E^{-1}\beta^{-\frac{1}{2}}\right\}.$$



Estimation of the consistency error $|||I^*_{\mathcal{M}} R_h(u_{\mathcal{I}})|||_*$

- $$|||I^*_{\mathcal{M}} R_h(u_{\mathcal{I}})|||_* = \sup_{v \in H^1_0(\Omega)} \frac{\langle R_h(u_{\mathcal{I}}) \, , \, I_{\mathcal{M}} v \rangle}{|||v|||} = \sup_{v \in H^1_0(\Omega)} \frac{S_n(u_{\mathcal{I}}, I_{\mathcal{M}} v)}{|||v|||}$$
- Streamline diffusion and interior penalty methods:
  $$|||I^*_{\mathcal{M}} R_h(u_{\mathcal{I}})|||_* \le c\eta^n_h$$
- Local projection scheme and subgrid-scale approach: $\nabla I_{\mathcal{M}} v \in \ker \kappa_{\mathcal{M}}$ and $I_{\mathcal{M}} v \in \ker \Pi_n$ hence
  $$|||I^*_{\mathcal{M}} R_h(u_{\mathcal{I}})|||_* = 0.$$

A Posteriori Error Estimate

- Define the space-time error estimator by:
  $$\eta^n = \tau_n^{\frac{1}{2}} \left[\underbrace{(\eta^n_h)^2}_{\text{spatial}} + \underbrace{\left\| u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}} \right\|^2 + (|||\Phi_{\mathcal{T}_n}||| + \eta^n_{\tau})^2}_{\text{temporal}} \right]^{\frac{1}{2}}.$$
- Then
  $$\|e\|_{X(0,T)} \le c^* \left\{ \|u_0 - \pi_0 u_0\|^2 + \sum_{n=1}^{N_{\mathcal{I}}} (\eta^n)^2 \right\}^{\frac{1}{2}},$$
  $$\eta^n \le c_* \|e\|_{X(t_{n-1},t_n)}.$$
- c_*, c^* only depend on the polynomial degrees and the shape parameters of the partitions $\widetilde{\mathcal{T}}_n$.

References

- L. Tobiska, R. Verfürth
  Robust a posteriori error estimates for stabilized finite element methods
  IMA J. Numer. Anal. **35** (2015), no. 4, 1652 – 1671
- H.-G. Roos, M. Stynes, and L. Tobiska
  Robust Numerical Methods for Singularly Perturbed Differential Equations
  Springer (2008)
- R. Verfürth
  A Posteriori Error Estimation Techniques for Finite Element Methods
  Oxford University Press (2013)