Robust A Posteriori Error Estimates for Stabilized FEM

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Overview

- There is a wide range of stabilized finite element methods for stationary and non-stationary convection-diffusion equations: streamline diffusion methods, local projection schemes, subgrid stabilization schemes, continuous interior penalty methods...
- Their a posteriori error analysis can be performed within a common framework.
- They all give rise to the same robust a posteriori error estimates up to slight variations in the data oscillation.
- Stationary convection-diffusion equations
- Non-stationary convection-diffusion equations
- References

Stationary Convection-Diffusion Equations

$$
\begin{aligned}
-\varepsilon\Delta u + \mathbf{a}\cdot\nabla u + bu &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \Gamma
\end{aligned}
$$

- $0 < \varepsilon \ll 1$
- $f \in L^2(\Omega)$, $\mathbf{a} \in W^{1,\infty}(\Omega)^d$, $b \in L^\infty(\Omega)$
- $-\frac{1}{2}\operatorname{div}\mathbf{a} + b \geq \beta$, $\|b\|_\infty \leq c_b\beta$

Variational Formulation

- $B(u,v) = \langle \ell\,,\, v\rangle \ \forall v \in H_0^1(\Omega)$
- $B(u,v) = \int_\Omega \{\varepsilon\nabla u\cdot\nabla v + \mathbf{a}\cdot\nabla u v + buv\}$, $\langle \ell\,,\, v\rangle = \int_\Omega fv$
- Energy norm: $|||v||| = \left\{\varepsilon\,\|\nabla v\|^2 + \beta\,\|v\|^2\right\}^{\frac{1}{2}}$
- Dual norm: $|||\varphi|||_* = \sup_v \frac{\langle \varphi\,,\, v\rangle}{|||v|||}$
- Continuity: $B(v,w) \leq \max\{c_b, 1\}\,\{|||v||| + |||\mathbf{a}\cdot\nabla v|||_*\}\,|||w|||$
- Stability:
$$\inf_v \sup_w \frac{B(v,w)}{\{|||v||| + |||\mathbf{a}\cdot\nabla v|||_*\}\,|||w|||} \geq \frac{1}{2+\max\{c_b,1\}}$$

Finite Element Spaces

- $\mathcal{T}$ admissible, affine-equivalent, shape-regular partition of Ω
- $S^{k,-1}(\mathcal{T}) = \{\varphi : \Omega \to \mathbb{R} : \varphi|_K \in R_k(K) \text{ for all } K \in \mathcal{T}\}$
  $S^{k,0}(\mathcal{T}) = S^{k,-1}(\mathcal{T}) \cap C(\overline{\Omega})$
  $S_0^{k,0}(\mathcal{T}) = \{\varphi \in S^{k,0}(\mathcal{T}) : \varphi = 0 \text{ on } \Gamma\}$
- $\mathcal{E}$ $(d-1)$-dimensional faces of elements, $\mathcal{E}_\Omega$ interior faces
- $\mathbb{J}_E(\cdot)$ jump across $E \in \mathcal{E}$ in direction $\mathbf{n}_E$
- $\mathcal{M}$ macro-partition subordinate to $\mathcal{T}$ with elements of comparable size

Stabilized FEM

- $B(u_\mathcal{T}, v_\mathcal{T}) + S_\mathcal{T}(u_\mathcal{T}, v_\mathcal{T}) = \langle \ell\,,\, v_\mathcal{T}\rangle \; \forall v_\mathcal{T} \in S_0^{k,0}(\mathcal{T})$
- Stabilization $S_\mathcal{T}$:
  - only depends on data ε, $\mathbf{a}$, b, f
  - is linear in its second argument
  - is affine in its first argument

Stabilization $S_\mathcal{T}$ I

- Streamline Diffusion Method (SDFEM): $S_\mathcal{T}(u_\mathcal{T}, v_\mathcal{T}) = \sum_{K \in \mathcal{T}} \vartheta_K \int_K \{-\varepsilon \Delta u_\mathcal{T} + \mathbf{a} \cdot \nabla u_\mathcal{T} + b u_\mathcal{T} - f\} \, \mathbf{a} \cdot \nabla v_\mathcal{T}$
- Local Projection Scheme (LPS):
  $S_\mathcal{T}(u_\mathcal{T}, v_\mathcal{T}) = \sum_{K \in \mathcal{T}} \vartheta_K \int_K \kappa_\mathcal{M} (\mathbf{a} \cdot \nabla u_\mathcal{T}) \, \kappa_\mathcal{M} (\mathbf{a} \cdot \nabla v_\mathcal{T})$
  $I - \kappa_\mathcal{M}$ L^2-projection onto $S^{k-1,-1}(\mathcal{M})$
- Subgrid Stabilization (SGS):
  $S_\mathcal{T}(u_\mathcal{T}, v_\mathcal{T}) = \sum_{K \in \mathcal{T}} \vartheta_K \int_K \nabla (\overline{\kappa}_\mathcal{M} u_\mathcal{T}) \cdot \nabla (\overline{\kappa}_\mathcal{M} v_\mathcal{T})$
  $\overline{\kappa}_\mathcal{M} = I - J_\mathcal{M}$, $J_\mathcal{M}$ quasi-interpolation operator in $S_0^{\ell,0}(\mathcal{M})$
- $\vartheta_K \|\mathbf{a}\|_{\infty;K} \lesssim h_K$

Stabilization $S_\mathcal{T}$ II

- Continuous Interior Penalty Method (CIP):
  $S_\mathcal{T}(u_\mathcal{T}, v_\mathcal{T}) = \sum_{E \in \mathcal{E}_\Omega} \vartheta_E \int_E \mathbb{J}_E(\mathbf{a} \cdot \nabla u_\mathcal{T}) \mathbb{J}_E(\mathbf{a} \cdot \nabla v_\mathcal{T})$
- $\vartheta_E \lesssim h_E^2$

Residual

- Residual: $\langle R\,,\,v\rangle = \langle \ell\,,\,v\rangle - B(u_{\mathcal{T}}, v)$
- Equivalence of error and residual: $|||R|||_* \approx |||u - u_{\mathcal{T}}||| + |||\mathbf{a}\cdot\nabla(u-u_{\mathcal{T}})|||_*$
- L^2-representation: $\langle R\,,\,v\rangle = \int_\Omega rv + \int_\Sigma jv$
  $r\big|_K = f + \varepsilon\Delta u_{\mathcal{T}} - \mathbf{a}\cdot\nabla u_{\mathcal{T}} - bu_{\mathcal{T}},\ j\big|_E = -\mathbb{J}_E(\varepsilon \mathbf{n}_E\cdot\nabla u_{\mathcal{T}})$
- Consistency error: $\langle R\,,\,v_{\mathcal{T}}\rangle = S_{\mathcal{T}}(u_{\mathcal{T}}, v_{\mathcal{T}})$
- Quasi-interpolation operator: $I_{\mathcal{M}} : H^1_D(\Omega) \to S^{1,0}_0(\mathcal{M})$
  $\|v - I_{\mathcal{M}}v\|_M \lesssim \|v\|_{\widetilde{\omega}_M},\ \|\nabla(v - I_{\mathcal{M}}v)\|_M \lesssim \|\nabla v\|_{\widetilde{\omega}_M}$
  $\|v - I_{\mathcal{M}}v\|_M \lesssim h_M\|\nabla v\|_{\widetilde{\omega}_M},\ \|v - I_{\mathcal{M}}v\|_F \lesssim h_F^{\frac12}\|\nabla v\|_{\widetilde{\omega}_F}$

Bounds for the Residual

- Error indicator:
  $$\eta_K = \left\{ \hbar_K{}^2\|r\|_K^2 + \frac12 \sum_{E\in\mathcal{E}_{K,\Omega}} \varepsilon^{-\frac12}\hbar_E\|j\|_E^2 \right\}^{\frac12}$$
- Data oscillation:
  $$\theta_K = \left\{ \hbar_K{}^2\|f - f_{\mathcal{T}} + (\mathbf{a}_{\mathcal{T}} - \mathbf{a})\cdot\nabla u_{\mathcal{T}} + (b_{\mathcal{T}} - b)u_{\mathcal{T}}\|_K^2 \right\}^{\frac12}$$
- $\hbar_\omega = \min\left\{\varepsilon^{-\frac12}\operatorname{diam}(\omega), \beta^{-\frac12}\right\}$
- Upper bound: $|||R|||_* \lesssim \left\{\sum_{K\in\mathcal{T}} \eta_K^2\right\}^{\frac12} + |||I^*_{\mathcal{M}}R|||_*$
- Lower bound: $\left\{\sum_{K\in\mathcal{T}} \eta_K^2\right\}^{\frac12} \lesssim \left[|||R|||_* + \left\{\sum_{K\in\mathcal{T}}\theta_K^2\right\}^{\frac12}\right]$

Consistency Error of SDFEM

- $|||I^*_{\mathcal{M}}R|||_* = \sup_v \dfrac{S_{\mathcal{T}}(u_{\mathcal{T}}, I_{\mathcal{M}}v)}{|||v|||}$
- $\sum_{K\in\mathcal{T}} \vartheta_K \int_K \{-\varepsilon\Delta u_{\mathcal{T}} + \mathbf{a}\cdot\nabla u_{\mathcal{T}} + bu_{\mathcal{T}} - f\}\,\mathbf{a}\cdot\nabla(I_{\mathcal{M}}v) \le$
  $\sum_{K\in\mathcal{T}} \vartheta_K\|r\|_K\|\mathbf{a}\cdot\nabla(I_{\mathcal{M}}v)\|_K$
- $\|\mathbf{a}\cdot\nabla(I_{\mathcal{M}}v)\|_K \le$
  $\|\mathbf{a}\|_{\infty;K}\min\{2c_3\varepsilon^{-\frac12}\|\nabla v\|_{\widetilde{\omega}_M}\ ,\ 2c_Ic_1h_M^{-1}\|v\|_{\widetilde{\omega}_M}\}$
- $|||I^*_{\mathcal{M}}R|||_* \lesssim \left\{\sum_{K\in\mathcal{T}}\eta_K^2\right\}^{\frac12}$

Consistency Error of LPS $(\sup_v \frac{S_{\mathcal{T}}(u_{\mathcal{T}}, I_{\mathcal{M}}v)}{|||v|||})$

- $\sum_{K\in\mathcal{T}} \vartheta_K \int_K \kappa_{\mathcal{M}}(\mathbf{a}\cdot\nabla u_{\mathcal{T}})\,\kappa_{\mathcal{M}}(\mathbf{a}\cdot\nabla(I_{\mathcal{M}}v))$
- $\mathbf{a}_{\mathcal{M}}\cdot\nabla u_{\mathcal{T}} \in S^{k-1,-1}(\mathcal{M})$ implies
  $\kappa_{\mathcal{M}}(\mathbf{a}\cdot\nabla u_{\mathcal{T}}) = \kappa_{\mathcal{M}}((\mathbf{a}-\mathbf{a}_{\mathcal{M}})\cdot\nabla u_{\mathcal{T}})$
- $\mathbf{a}_{\mathcal{M}}\cdot\nabla(I_{\mathcal{M}}v) \in S^{0,-1}(\mathcal{M}) \subset S^{k-1,-1}(\mathcal{M})$ implies
  $\kappa_{\mathcal{M}}(\mathbf{a}\cdot\nabla(I_{\mathcal{M}}v)) = \kappa_{\mathcal{M}}((\mathbf{a}-\mathbf{a}_{\mathcal{M}})\cdot\nabla(I_{\mathcal{M}}v))$
- $|||I^*_{\mathcal{M}}R|||_* \lesssim \left\{\sum_{K\in\mathcal{T}} h_K^2\|\nabla\mathbf{a}\|^2_{\infty;K}\theta_K^2\right\}^{\frac12}$

Consistency Error of SGS $\left(\sup_v \frac{S_{\mathcal{T}}(u_{\mathcal{T}}, I_{\mathcal{M}} v)}{\|v\|}\right)$

- $\sum_{K \in \mathcal{T}} \vartheta_K \int_K \nabla(\overline{\kappa}_{\mathcal{M}} u_{\mathcal{T}}) \cdot \nabla(\overline{\kappa}_{\mathcal{M}}(I_{\mathcal{M}} v))$
- $\overline{\kappa}_{\mathcal{M}} = I - J_{\mathcal{M}}$
- Assume that $J_{\mathcal{M}}$ reproduces $S_0^{1,0}(\mathcal{M})$.
- $|||I_{\mathcal{M}}^* R|||_* = 0$

Consistency Error of CIP $\left(\sup_v \frac{S_{\mathcal{T}}(u_{\mathcal{T}}, I_{\mathcal{M}} v)}{\|v\|}\right)$

- $\sum_{E \in \mathcal{E}_\Omega} \vartheta_E \int_E \mathbb{J}_E(\mathbf{a} \cdot \nabla u_{\mathcal{T}}) \mathbb{J}_E(\mathbf{a} \cdot \nabla(I_{\mathcal{M}} v))$
- Express $\mathbb{J}_E(\mathbf{a} \cdot \nabla u_{\mathcal{T}})$ in terms of residual r and data oscillations using continuous approximations for $\mathbf{a}$ and b.
- $|||I_{\mathcal{M}}^* R|||_* \lesssim \left\{ \sum_{K \in \mathcal{T}} \left[\eta_K^2 + \theta_K^2 + h_K^2 \hbar_K^2 \|\nabla \mathbf{a}\|_{\infty;K}^2 \|\nabla u_{\mathcal{T}}\|_K^2 + \varepsilon^2 \hbar_K^2 \|\Delta u_{\mathcal{T}}\|_K^2 \right] \right\}^{\frac{1}{2}}$

Non-Stationary Convection-Diffusion Equations

$$
\begin{aligned}
\partial_t u - \varepsilon \Delta u + \mathbf{a} \cdot \nabla u + bu = f &\quad \text{in } \Omega \times (0,T] \\
u = 0 &\quad \text{on } \Gamma \times (0,T] \\
u(\cdot, 0) = u_0 &\quad \text{in } \Omega
\end{aligned}
$$

- $0 < \varepsilon \ll 1$
- $f \in L^2(\Omega)$, $\mathbf{a} \in W^{1,\infty}(\Omega)^d$, $b \in L^\infty(\Omega)$ constant w.r.t. t
- $-\frac{1}{2} \operatorname{div} \mathbf{a} + b \geq \beta$, $\|b\|_\infty \leq c_b \beta$

Variational Formulation

- $\langle \partial_t u , v \rangle + B(u, v) = \langle \ell , v \rangle$ $\forall t \in (0,T), v \in H_0^1(\Omega)$
- $\|v\|_{L^\infty(a,b;L^2)} = \operatorname{ess.sup}_{a<t<b} \|v(\cdot, t)\|$
- $\|v\|_{L^2(a,b;H^1)} = \left\{ \int_a^b |||v(\cdot,t)|||^2 \, dt \right\}^{\frac{1}{2}}$
- $\|v\|_{L^2(a,b;H^{-1})} = \left\{ \int_a^b |||v(\cdot,t)|||_*^2 \, dt \right\}^{\frac{1}{2}}$

Stabilized FEM

- $\mathcal{I} = \{[t_{n-1}, t_n] : 1 \le n \le N_{\mathcal{I}}\}$ partition of $[0, T]$
- $\mathcal{T}_n$ partitions of Ω as in the stationary case
- $u^0_{\mathcal{T}_0}$ L^2-projection of u_0
- $\frac{1}{\tau_n}\left(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}, v_{\mathcal{T}_n}\right) + B(U^{n\theta}, v_{\mathcal{T}_n}) + S_{\mathcal{T}_n}(U^{n\theta}, v_{\mathcal{T}_n}) = \langle \ell, v_{\mathcal{T}_n}\rangle \; \forall n \ge 1, v_{\mathcal{T}_n} \in S_0^{k,0}(\mathcal{T}_n)$
- $U^{n\theta} = \theta u^n_{\mathcal{T}_n} + (1-\theta) u^{n-1}_{\mathcal{T}_{n-1}}$
- B, ℓ, $S_{\mathcal{T}_n}$ as in the stationary case
- $\theta = \frac{1}{2}$ Crank-Nicolson scheme, $\theta = 1$ implicit Euler scheme
- $u_{\mathcal{I}}$ affine interpolation w.r.t. t of $\left(u^n_{\mathcal{T}_n}\right)_n$

Residual

- Residual: $\langle R, v\rangle = \langle \ell, v\rangle - \langle \partial_t u_{\mathcal{I}}, v\rangle - B(u_{\mathcal{I}}, v)$
- Equivalence of error and residual:
  $$\|R\|_{L^2(0,T;H^{-1})} \approx \Big\{\|u - u_{\mathcal{I}}\|^2_{L^\infty(0,T;L^2)} + \|u - u_{\mathcal{I}}\|^2_{L^2(0,T;H^1)} + \|\partial_t(u - u_{\mathcal{I}}) + \mathbf{a}\cdot\nabla(u - u_{\mathcal{I}})\|^2_{L^2(0,T;H^{-1})}\Big\}^{\frac{1}{2}}$$
- Decomposition of the residual: $R = R_\tau + R_h$
  $\langle R_\tau, v\rangle = B(U^{n\theta} - u_{\mathcal{I}}, v)$
  $\langle R_h, v\rangle = \langle \ell, v\rangle - \langle \partial_t u_{\mathcal{I}}, v\rangle - B(U^{n\theta}, v)$
  $\|R\|^2_{L^2(0,T;H^{-1})} \approx \|R_\tau\|^2_{L^2(0,T;H^{-1})} + \|R_h\|^2_{L^2(0,T;H^{-1})}$
- Consistency error: $\langle R_h, v_{\mathcal{T}_n}\rangle = S_{\mathcal{T}_n}(U^{n\theta}, v_{\mathcal{T}_n})$

Bounds for the Residuals

- Temporal residual R_τ: As for non-stabilized schemes
  $\sqrt{\tau_n}\left\{\left|\!\left|\!\left|u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}\right|\!\right|\!\right| + \left|\!\left|\!\left|\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})\right|\!\right|\!\right|_*\right\} \approx \|R_\tau\|_{L^2(t_{n-1},t_n;H^{-1})}$
- Convective derivative: As for non-stabilized schemes solving an auxiliary discrete reaction diffusion equation.
- Spatial residual R_h: As for stabilized schemes in the stationary case.

References

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